Prediction Market Prices under Risk Aversion and Heterogeneous Beliefs

Xue-Zhong He* Nicolas Treich†

Abstract

In this paper, we examine the properties of prediction market prices when risk averse traders have heterogeneous beliefs in state probabilities. We show that the equilibrium state prices equal the mean beliefs of traders about that state if and only if the traders' common utility function is logarithmic. We also provide a necessary and sufficient condition ensuring that the state prices are systematically below or above the mean beliefs of traders, thus providing a rational explanation to the favorite-longshot bias in prediction markets.

Keywords: Prediction market, heterogeneous beliefs, risk aversion, favorite-longshot bias.

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*University of Technology Sydney, Business School, Finance Discipline Group, Australia. Email: tony.he1@uts.edu.au

†Toulouse School of Economics, INRA, University Toulouse Capitole, France. Corresponding author. Email: ntreich@toulouse.inra.fr

1 Introduction

1.1 General motivation

Prediction markets are one of the most efficient tools to elicit people's beliefs (Wolfers and Zitzewitz 2004, Surowiecki 2005, Hahn and Tetlock 2006). They have been repeatedly used to predict the outcome of political elections, like with the Iowa electronic market. They are also increasingly used by private companies to elicit their employees' beliefs about future sales or industry trends. Technically, prediction markets are simple financial markets in which traders bet on the outcomes of uncertain events. Asset prices in prediction markets are typically interpreted as probabilities. For instance, Arrow et al. (2008) introduce prediction markets as follows: *"Consider a contract that pays $1 if Candidate X wins the presidential election in 2008. If the market price of an X contract is currently 53 cents, an interpretation is that the market 'believes' X has a 53% chance of winning"* (Arrow et al. 2008).

Given the development of prediction markets, it is crucial to understand theoretically what prediction market prices exactly predict. In particular, it is important to understand the relationship between these prices and the probabilistic beliefs of traders about future events. In this paper, we examine theoretically one specific question about this relationship, namely under which conditions prediction market prices equal the mean of traders' beliefs. In other words, we study when a prediction market is, or is not, well calibrated on average (Page and Clemen 2013).

1.2 Main results

We consider a two-state prediction market model in which traders have heterogeneous prior beliefs about future states (Gjerstad 2004, Manski 2006, Wolfers and Zitzewitz 2006). We obtain two main results. Firstly, we derive the exact necessary and sufficient condition for prediction market state prices to be equal to the mean beliefs of traders about that state. This condition holds for all possible distribution of beliefs *if and only if* the traders' common utility function is logarithmic. We also provide a sufficient condition on the distribution of beliefs for all possible common utility functions. Secondly, when prediction market prices are different from the mean beliefs, we characterize the determinants of that difference. Most significantly, we exhibit the necessary and sufficient condition for the equilibrium price to be always below or always above the mean beliefs for all symmetric belief distributions. More precisely, there exists a *favorite-longshot bias*, meaning that the high-likelihood events are underpriced and the low-likelihood events are overpriced, *if and only if* the traders' risk preferences are such that twice absolute risk aversion is less than absolute prudence. Under constant relative risk aversion (CRRA) γ this condition is equivalent to $\gamma < 1$. That is, when traders' utility is less risk averse than the logarithmic utility, we provide a rationale to the well documented favorite-longshot bias (Ali 1977, Thaler and Ziemba 1988). In a discussion on the extension to a many-state prediction market, we show that the equilibrium state price not only depends on the distribution of beliefs about that state, but also about the other states.

1.3 Related literature

Several papers have studied trade in a prediction market by a population of individuals with heterogeneous prior beliefs, including Gjerstad (2004), Manski (2006) and Wolfers and Zitzewitz (2006). When traders are risk neutral and have limited investment budgets, Manski (2006) shows that the equilibrium price of the prediction market differs in general from the mean beliefs of traders. When traders are risk averse, Wolfers and Zitzewitz (2006) show theoretically that the prediction market price equals the mean of traders' beliefs when the common utility function of traders is logarithmic. Moreover, Wolfers and Zitzewitz (2006) explore numerically how the equilibrium price is affected by belief heterogeneity for several utility functions and several belief distributions.[1] In this paper, we show that the logarithmic utility is not only sufficient but also necessary to ensure the prediction market prices equal the mean beliefs of traders for all belief distributions. More

[1]See Gjerstad (2004) for theoretical results under CRRA utility functions, and some numerical results. See also Fountain and Harrison (2011) for further numerical results with wealth and belief heterogeneity.

interestingly, we provide the necessary and sufficient condition for the equilibrium prices to be systemically biased for all symmetric belief distributions. This condition depends critically on the risk preferences and the mean beliefs. It provides a theoretical foundation to the numerical results in Wolfers and Zitzewitz (2006) and a complete characterization of the theoretical results in Gjerstad (2004).

Our paper is also related to some recent papers providing an informational explanation to the favorite-longshot bias (Ottaviani and Sorensen 2009, 2010, 2015). Ottaviani and Sorensen (2009) assume that traders have a common prior but incorporate in a Bayesian fashion the information revealed by the bets placed at the equilibrium. This model is generalized in Ottaviani and Sorensen (2010) to allow for noise and private information. Recently, Ottaviani and Sorensen (2015) study the price underreaction when traders with heterogeneous beliefs react to public information. In all these models, Ottaviani and Sorensen derive a sufficient condition leading to the favorite-longshot bias in a binary prediction market, which is different from the necessary and sufficient condition presented in this paper. Finally, we recall that a prediction market is a simple financial market. Therefore, this paper is also closely related to the models of asset pricing under heterogeneous beliefs (see, e.g., Varian 1985, Abel 1989, Jouini and Napp 2006 and 2007, Gollier 2007 and Roche 2011). These models have been used to explain various anomalies in financial markets.[2]

1.4 Outline

We organize the paper as follows. In the next section we introduce a simple two-state prediction market model and derive a sufficient condition for the equilibrium state price to exist and to be unique. In Section 3, we derive the necessary and sufficient condition for prediction market equilibrium prices to be equal to the mean beliefs of traders. Then in Section 4 we examine the conditions leading to a favorite-longshot bias. Finally in Section 5 we provide a discussion on the generalization of the previous results to a prediction

[2]For a justification and implications of these models see, for instance, the literature survey papers by Varian (1989), Scheinkman and Xiong (2004) and Hong and Stein (2007).

market with more than two states. The last section concludes. The proofs of the results presented in Section 5 are given in the Appendix.

2 The basic model and its equilibrium properties

We consider a simple prediction market in which traders can buy and sell a risky asset paying \$1 if a specific state occurs, and nothing otherwise. The basic model relies on three important assumptions: (i) there are only two states (i.e., an event occurs or not); (ii) the traders' prior beliefs are heterogeneous; and (iii) the traders have the same utility function and the same wealth.

Assumption (i) is a standard assumption in prediction market models; moreover, it is relaxed in Section 5 where we study a general prediction market with many states. Assumption (ii) is the key assumption of the model. It means that traders "agree to disagree", and therefore have different prior beliefs about the states. Namely, the heterogeneity in beliefs does not come from asymmetric information but rather from intrinsic differences in how traders interpret information. This assumption has been common in economic models of prediction markets cited in the previous section. Moreover, to focus on the specific effect of belief heterogeneity, we assume (iii), that the utility function and the wealth of traders are homogeneous.[3] In the following, we derive the equilibrium properties of this basic model.

Formally, when a trader decides how much to invest in the financial asset paying \$1 if an event occurs, he maximizes the following expected utility:

$$\max_{\alpha}[pu(w+\alpha(1-\pi))+(1-p)u(w-\alpha\pi)], \tag{1}$$

in which w is his initial wealth, $p \in (0,1)$ his subjective probability or belief that the event occurs, α his asset demand and π the price of this asset.[4] We assume that the trader's vNM utility function $u(\cdot)$ is strictly increasing, strictly concave and three times

[3]See Remark 1 and Example 2 in the next section for a discussion on the affect of heterogeneity in wealth and the utility function of traders.

[4]The individual asset demand α can be seen as the net asset demand of one asset in a model with

differentiable. We first explore the properties of the individual asset demand.

Lemma 1. *For a given probability p and asset price π, the asset demand of the trader $\alpha(p,\pi) \geq 0$ if and only if $p \geq \pi$; namely, the trader buys (respectively sells) the asset yielding \$1 when the event occurs and 0 otherwise if and only if he assigns a probability for this event higher (respectively lower) than the asset price.*

Proof: With the objective function (1), the first order condition is given by

$$p(1-\pi)u'(w+\alpha(p,\pi)(1-\pi)) - (1-p)\pi u'(w-\alpha(p,\pi)\pi) = 0, \tag{2}$$

in which $\alpha(p,\pi)$ is the solution of (2). Differentiating equation (2) with respect to p,

$$\begin{aligned} 0 \;=\; & (1-\pi)u'(w+\alpha(p,\pi)(1-\pi)) + \pi u'(w-\alpha(p,\pi)\pi) \\ & +\alpha_p(p,\pi)[p(1-\pi)^2 u''(w+\alpha(p,\pi)(1-\pi)) \\ & \qquad +(1-p)\pi^2 u''(w-\alpha(p,\pi)\pi)], \end{aligned} \tag{3}$$

we obtain

$$\alpha_p(p,\pi) = \frac{(1-\pi)u'(w+\alpha(p,\pi)(1-\pi)) + \pi u'(w-\alpha(p,\pi)\pi)}{-p(1-\pi)^2 u''(w+\alpha(p,\pi)(1-\pi)) - (1-p)\pi^2 u''(w-\alpha(p,\pi)\pi)} > 0, \tag{4}$$

that is, the asset demand strictly increases with belief p. Since $\alpha(p,p) = 0$, we conclude that $\alpha(p,\pi) \geq 0$ if and only if $p \geq \pi$.■

We now introduce the prediction market with N traders, indexed by $i = 1, ..., N$. Each trader has a belief p_i and the *distribution* of beliefs in the population of traders is denoted $\widetilde{p} \equiv (p_1, ..., p_N; 1/N, ..., 1/N)$. Heterogeneity in p_i is the only source of heterogeneity in our basic model, so that each trader i's demand for a given price π can be denoted $\alpha(p_i,\pi)$.

two Arrow-Debreu assets. To see that, let α_s and π_s denote respectively the demand for and the price of Arrow-Debreu assets in state $s = 1, 2$. The objective can then be written: $\max_{\alpha_1,\alpha_2}[pu(w+\alpha_1-\pi_1\alpha_1-\pi_2\alpha_2)+(1-p)u(w+\alpha_2-\pi_1\alpha_1-\pi_2\alpha_2)]$. Denoting $\alpha = \alpha_1 - \alpha_2$ and observing that $\pi_1 + \pi_2 = 1$ by arbitrage then leads (with $\pi = \pi_1$) to (1).

Let π^* be the equilibrium price, the equilibrium condition can thus be written

$$\sum_{i=1}^{N} \alpha(p_i, \pi^*) = 0. \qquad (5)$$

Let $\overline{p} = (1/N)\sum_{i=1}^{N} p_i$ denote the mean beliefs of traders. Our main objective in this paper is to compare π^* to $\overline{p}$. In particular, in Section 3 we derive conditions so that $\pi^* = \overline{p}$. Since $\alpha(p,p) = 0$, notice immediately that when the belief distribution $\widetilde{p}$ is degenerate (i.e., $p_i = p$ for all i), then $\pi^* = p$ and there is no trade at the equilibrium. We rule out this homogeneous beliefs case (until Section 5), and consider a nondegenerate distribution $\widetilde{p}$ in the following. As another preliminary result, we show that an equilibrium always exists in our basic prediction market model.

Lemma 2. *There always exists an equilibrium price π^*.*

Proof: Following Lemma 1, we have $\alpha(p, \pi) > 0$ if and only if $p > \pi$. Therefore, when π tends to 0 (respectively tends to 1) $\alpha(p, \pi)$ becomes positive (respectively negative) for all p, so that $\sum_{i=1}^{N} \alpha(p_i, \pi)$ also becomes positive (respectively negative). This implies that when π increases, $\sum_{i=1}^{N} \alpha(p_i, \pi)$ as a function of π must go from a positive to a negative region and thus crosses zero somewhere in between at least once. As a result, there must be a solution to (5).■

We now discuss the uniqueness of the equilibrium. This holds if $\sum_{i=1}^{N} \alpha(p_i, \pi)$ as a function of π only crosses the origin once. We know that $\alpha(p, \pi)$ has this single crossing property at $\pi = p$. But that does not guarantee that $\sum_{i=1}^{N} \alpha(p_i, \pi)$ also has the single crossing property and thus has a unique equilibrium, as illustrated by the following example.

Example 1 (Multiple equilibria): Consider traders with a quadratic utility function $u(w) = -(1-w)^2$ for $0 \le w \le 1$ and initial wealth $w = 1/2$. Following the first order condition the optimal asset demand is equal to $\alpha(p, \pi) = \frac{p-\pi}{2(p-2p\pi+\pi^2)}$. In a prediction market with only two traders with respective beliefs of the event denoted $p_1 = 0.1$ and $p_2 = 0.9$, the equilibrium condition is equivalent to $9 - 68\pi + 150\pi^2 - 100\pi^3 = 0$. Solving for this equation, it is found that there are three equilibrium prices in this prediction market: $\pi^* = (0.235, 0.5, 0.764)$.

We now provide a sufficient condition for uniqueness under decreasing absolute risk aversion (DARA).

Proposition 1 *The equilibrium price π^* is unique if u has decreasing absolute risk aversion (DARA).*

Proof: The equilibrium is unique if $\sum_{i=1}^{N} \alpha(p_i, \pi)$ is a strictly decreasing function of π. Therefore, a sufficient condition for the uniqueness of the equilibrium is $\alpha_\pi(p, \pi) < 0$ everywhere. Differentiating (2) with respect to π, we have

$$\alpha_\pi(p,\pi) = \frac{-pu'(w+\alpha(p,\pi)(1-\pi)) - (1-p)u'(w-\alpha(p,\pi)\pi)}{-p(1-\pi)^2u''(w+\alpha(p,\pi)(1-\pi)) - (1-p)\pi^2u''(w-\alpha(p,\pi)\pi)} \tag{6}$$
$$-\alpha(p,\pi)\frac{p(1-\pi)u''(w+\alpha(p,\pi)(1-\pi)) - (1-p)\pi u''(w-\alpha(p,\pi)\pi)}{-p(1-\pi)^2u''(w+\alpha(p,\pi)(1-\pi)) - (1-p)\pi^2u''(w-\alpha(p,\pi)\pi)}.$$

The first term is strictly negative but the second term is of ambiguous sign, so that the demand may increase when the price π increases, as it is the case in Example 1. When u has DARA, we can show that the second term of the right hand side of (6) is negative.

Let $\widetilde{x} \equiv (1-\pi, -\pi; p, 1-p)$ denote a random variable $\widetilde{x}$ which takes values of $1-\pi$ and $-\pi$ with probabilities p and $1-p$, respectively. The first order condition (2) can thus be written more compactly as

$$E[\widetilde{x}u'(w+\alpha\widetilde{x})] = 0. \tag{7}$$

To show that the second term of the right hand side of (6) is negative, we thus need to show that $-\alpha E[\widetilde{x}u''(w+\alpha\widetilde{x})] \leq 0$ with α defined in (7). Now let us introduce a concave function T so that T' is decreasing, implying

$$\alpha x T'(u(w+\alpha x)) \leq \alpha x T'(u(w)),$$

for all α and x. Multiplying by $u'(w+\alpha x)$ and taking expectations, this implies

$$\alpha E[\widetilde{x}T'(u(w+\alpha\widetilde{x}))u'(w+\alpha\widetilde{x})] \leq \alpha T'(u(w))E[\widetilde{x}u'(w+\alpha\widetilde{x})] = 0.$$

Finally, let us pick T so that $-u' = T(u)$. We in turn have $-u'' = T'(u)u'$ which, by using the last inequality, leads to $-\alpha E[\widetilde{x}u''(w+\alpha\widetilde{x})] \leq 0$. We thus have shown that the result holds whenever $-u'$ is more risk averse than u, which is equivalent to decreasing absolute risk aversion.■

This result is consistent with Proposition 4 of Ottaviani and Sorensen (2015), although it is obtained in a different setting here. The intuition for this result is the following. When the price of an asset increases, there are two effects captured by the two terms on the right hand side of equation (6). First, there is a substitution effect that leads to a decrease in its demand, but there is also a wealth effect that may potentially increase its demand. Intuitively, as the wealth distribution deteriorates, the investor's attitude towards risk may change, and this wealth effect might prove sufficiently strong to increase the demand for the risky asset, as initially shown by Fishburn and Porter (1976) in the case of a first-order stochastic dominance (FSD) shift. Under DARA however, the negative wealth effect leads the trader to be more risk averse, and therefore further decreases the demand for the risky asset. Under constant absolute risk aversion (CARA), there is no wealth effect, and only the first negative effect is at play. Finally, we note that Example 1 features multiple equilibria because the quadratic utility function has increasing absolute risk aversion. Finally, we observe that, when there is a unique equilibrium, one can characterize the effect of a change in the distribution of beliefs on the equilibrium price. Indeed, from the equilibrium condition $\sum_{i=1}^{N} \alpha(p_i, \pi) = 0$ and $\alpha_p(p, \pi) > 0$, any FSD improvement in the distribution of beliefs must increase the equilibrium price.

3 Prediction market prices and the mean beliefs of traders

In this section, we first ask the following question: which utility functions lead the prediction market prices to equal the mean beliefs of traders, i.e. $\pi^* = \overline{p}$, for any belief distribution? To answer that question, we first consider a logarithmic utility function $u(w) = \log w$. Note that this function displays DARA, and thus the equilibrium is unique from Proposition 1. In that case, we can obtain a closed-form solution to the first-order condition (2):

$$\alpha(p, \pi) = w\frac{(p - \pi)}{\pi(1 - \pi)}.$$

This implies that the equilibrium condition (5) can simply be written $\pi^* = \overline{p}$. This shows that the logarithmic utility function is a sufficient condition, a result already obtained in Gjerstad (2004) and Wolfers and Zitzewitz (2006). A natural question is whether the utility function must be logarithmic to guarantee $\pi^* = \overline{p}$ or whether this result can arise for other utility functions. That is, we want to know whether $u(w) = \log w$ is also a necessary condition. We next show that this is indeed the case.

Proposition 2 *The prediction market price is equal to the mean beliefs of traders, i.e. $\pi^* = \overline{p}$, for all traders' belief distributions $\widetilde{p}$ if and only if $u(w) = \log w$ for all w.*

Proof: We just need to prove the necessity. Namely, we need to show that if $\sum_{i=1}^{N} \alpha(p_i, \overline{p}) = 0$ for all $\widetilde{p}$, then $u(w) = \log w$. Since $\sum_{i=1}^{N} \alpha(p_i, \overline{p}) = 0$ for all the heterogeneous beliefs $\widetilde{p}$, this must also hold for a specific case of symmetric heterogeneous beliefs with two traders having $p_1 = p + \delta$ and $p_2 = p - \delta$ for $p \neq 1/2$ and $\delta \in [0, \min\{p, 1-p\}]$. If $\pi^* = (p_1 + p_2)/2 = p$ is the equilibrium price, we must have

$$g(\delta) := \alpha(p + \delta, p) + \alpha(p - \delta, p) = 0,$$

in which $\alpha(p + \delta, p)$ is the unique solution of

$$(p + \delta)(1 - p)u'(w + \alpha(p + \delta, p)(1 - p)) - (1 - p - \delta)pu'(w - \alpha(p + \delta, p)p) = 0 \qquad (8)$$

and $\alpha(p-\delta, p)$ is the unique solution of

$$(p-\delta)(1-p)u'(w+\alpha(p-\delta,p)(1-p)) - (1-p+\delta)pu'(w-\alpha(p-\delta,p)p) = 0. \qquad (9)$$

Since $g(\delta) = 0$, we have $g(\delta) = g'(\delta) = g''(\delta) = 0$. Moreover, we have $g''(0) = 2\alpha_{pp}(p,p) = 0$. Differentiating again (3) with respect to p to compute $\alpha_{pp}(p,p)$ we obtain

$$\begin{aligned} 0 &= 2\alpha_p(p,\pi)\{(1-\pi)^2u''(w+\alpha(p,\pi)(1-\pi)) - \pi^2u''(w-\alpha(p,\pi)\pi)\} \\ &+ \alpha_{pp}(p,\pi)\{p(1-\pi)^2u''(w+\alpha(p,\pi)(1-\pi)) + (1-p)\pi^2u''(w-\alpha(p,\pi)\pi)\} \\ &+ \alpha_p(p,\pi)^2\{p(1-\pi)^3u'''(w+\alpha(p,\pi)(1-\pi)) - (1-p)\pi^3u'''(w-\alpha(p,\pi)\pi)\}. \end{aligned}$$

Taking $\pi = p$ in the last expression, we have from (4) that

$$\alpha_p(p,p) = \frac{1}{p(1-p)} \times \frac{u'(w)}{-u''(w)}.$$

Then rearranging yields

$$\alpha_{pp}(p,p) = \frac{(1-2p)}{p^2(1-p)^2}\left[\frac{u'(w)}{u''(w)}\right]^2\left[\frac{u'''(w)}{-u''(w)} - 2\frac{-u''(w)}{u'(w)}\right]. \qquad (10)$$

Therefore $g''(0) = 0$ is equivalent to

$$(\frac{1}{2} - p)\left[\frac{u'''(w)}{-u''(w)} - 2\frac{-u''(w)}{u'(w)}\right] = 0$$

for all w. Since $p \neq 1/2$, we have $\frac{u'''(w)}{-u''(w)} = 2\frac{-u''(w)}{u'(w)}$. Finally, solving for this differential equation gives $u(w) = c_1 + c_2\log(w + c_3)$ where c_k, $k = 1, 2, 3$, are constants. Since $u(w)$ is defined up to a positive affine transformation, and since w is arbitrary, this is equivalent to $u(w) = \log w$.■

This result shows that the prediction market price is equal to the mean beliefs of traders for all possible distributions of beliefs. This is an important result. Unfortunately, the following remarks indicate that this result does not hold in more general settings in which wealth is either heterogenous or can vary with the event.

Remark 1 (Wealth heterogeneity): The result of Proposition 2 cannot be generalized to non-common wealth, as possible correlation between wealth and beliefs would invalidate the result. Indeed, let $\widetilde{w}$ be the distribution representing traders' wealth heterogeneity. Assuming a logarithmic utility function, we can obtain[5]

$$\pi^* = \overline{p} + \frac{1}{E_{\widetilde{w}}(\widetilde{w})} Cov(\widetilde{p}, \widetilde{w}). \tag{11}$$

Therefore there is no utility function that can always ensure prediction markets are unbiased when beliefs and wealth are potentially correlated. Observe that, despite this impossibility result, the direction of the bias can be inferred if the analyst knows the sign of the correlation between beliefs and wealth. The intuition for (11) is that richer individuals invest more, and therefore have more influence on the equilibrium price. Thus, if wealth is positively (respectively negatively) correlated with beliefs, the equilibrium price will be higher (respectively lower).

Remark 2 (Stakes): Suppose each trader has a (positive or negative) stake Δ in the event he predicts, so that he now maximizes over α the following expected utility

$$pu(w + \Delta + \alpha(1 - \pi)) + (1 - p)u(w - \alpha\pi).$$

Then it is easy to understand that the result of Proposition 2 is not guaranteed either. Indeed for the logarithmic utility function we have

$$\alpha(p, \pi) = w\frac{(p - \pi)}{\pi(1 - \pi)} - \Delta\frac{\pi(1 - p)}{\pi(1 - \pi)},$$

leading to the equilibrium condition

$$\pi^* = \frac{w\overline{p}}{w + \Delta(1 - \overline{p})}.$$

The intuition is that when there is a positive (respectively negative) stake, the marginal utility decreases (respectively increases) if the event occurs. As a result, the traders want

[5] Here $E_{\widetilde{w}}(\widetilde{w})$ denotes the traders' mean wealth and $Cov(\widetilde{p}, \widetilde{w})$ denotes the covariance between the traders' distributions of wealth and of beliefs.

to transfer wealth to the state in which the event does not occur (respectively occurs), and they typically use the prediction market as a hedging scheme to do this. The consequence is that the equilibrium is biased downwards (respectively upwards). Observe that if the stakes are individual-dependent but uncorrelated with beliefs, and if their mean across individuals is equal to zero, then we retrieve a prediction market that is unbiased under a logarithmic utility function.

The previous proposition provides the condition on the utility function so that $\pi^* = \overline{p}$ for all belief distributions $\widetilde{p}$. We now study the dual problem: what are the conditions on the belief distribution $\widetilde{p}$ leading to $\pi^* = \overline{p}$ for all utility functions u? We show that the answer depends on whether the distribution of beliefs is symmetric about one half.

Proposition 3 *If $\widetilde{p}$ is symmetric about $1/2$, then the prediction market price is equal to the mean beliefs of traders, i.e. $\pi^* = \overline{p}$, for all u that imply a unique equilibrium.*

Proof: We want to show that if $\widetilde{p}$ is symmetric about $1/2$ then $\pi^* = \overline{p}$ for all u. Observe from the first order condition (2) that $\alpha(p, \pi) = -\alpha(1-p, 1-\pi)$. This implies that the equilibrium condition can be written $\sum_{i=1}^{N} \alpha(p_i, \pi^*) = \sum_{i=1}^{N} \alpha(1-p_i, 1-\pi^*) = 0$. Observe then that $\widetilde{p}$ symmetric about $1/2$ means that $\widetilde{p}$ is distributed as $1-\widetilde{p}$. Consequently the equilibrium condition implies $\sum_{i=1}^{N} \alpha(p_i, \pi^*) = \sum_{i=1}^{N} \alpha(p_i, 1-\pi^*)$. Since the equilibrium is assumed to be unique, this last condition implies $\pi^* = 1 - \pi^*$, that is $\pi^* = 1/2 = \overline{p}$.$\blacksquare$

The intuition for Proposition 3 is simple. When $\widetilde{p}$ is symmetric about one half, the two states are formally indistinguishable. Therefore it cannot be that the price of an asset yielding one dollar in one state is different from that of an asset yielding one dollar in the other state, implying $\pi^* = 1/2$.

We note, however, that if heterogeneity in individual utility functions is introduced, then the result may not hold even when $\widetilde{p}$ is symmetric about $1/2$. The intuition is essentially the same as the one presented in Remark 1. This is illustrated by the following example which considers heterogeneity over (constant absolute) risk aversion.

Example 2 (Heterogeneous CARA): Let $u_i(w) = -e^{-r_i w}$ in which $r_i > 0$ represents the CARA coefficient of trader $i = 1, 2$ with respective beliefs $p_1 = 0.1$ and $p_2 = 0.9$ of the event. Under positive correlation between beliefs and risk aversion $(r_1, r_2) = (1, 3)$, we have $\pi^* = 1/4 < 1/2 = \overline{p}$, while under negative correlation $(r_1, r_2) = (3, 1)$, we have $\pi^* = 3/4 > 1/2 = \overline{p}$.

We have characterized the conditions leading to $\pi^* = \overline{p}$ for all belief distributions $\widetilde{p}$ in Proposition 2 and for all utility functions u in Proposition 3. These conditions are rather stringent. We note, however, that one can relax them in the sense that it is possible to find well-chosen pairs $(u, \widetilde{p})$ also leading to $\pi^* = \overline{p}$. This is shown in the following example which uses a specific CRRA utility function and a specific nonsymmetric distribution of beliefs.

Example 3 (Unbiased prediction market under CRRA and nonsymmetric beliefs). Consider traders with utility function $u(w) = -1/w$. Two groups of traders participate in the prediction market: one group has beliefs $p_1 = p$, and the other group has beliefs $p_2 = 1-p$. Denoting a the proportion of traders in the first group, we have $\overline{p} = ap + (1-a)(1-p)$. Note that the belief distribution among traders may not be symmetric about 1/2. One may then easily obtain that $\sum_{i=1}^{N} \alpha(p_i, \pi^*) = a\alpha(p, \pi^*) + (1-a)\alpha(1-p, \pi^*) = 0$ is equivalent to $\sqrt{\pi^*(1-\pi^*)}\{ap + (1-a)(1-p) - \pi^*\} = 0$ leading to $\pi^* = \overline{p}$.

Examples 2 and 3 indicate that the condition that the distribution of beliefs is symmetric about 1/2 is neither necessary nor sufficient to obtain $\pi^* = \overline{p}$ in general.

4 A necessary and sufficient condition for the favorite-longshot bias

In the previous analysis, we have examined under which conditions the prediction market equilibrium price π^* is equal to the mean beliefs $\overline{p}$ of traders. We have seen that these conditions are rather stringent, implying that a difference between the equilibrium price and the mean beliefs of traders is expected in prediction markets in general. It is therefore interesting to better characterize this difference, or "bias". In this section, we derive a necessary and sufficient condition so that the equilibrium price π^* is systematically above or below the mean belief $\overline{p}$.

The analysis developed in this section may provide a rationale for the favorite-longshot bias, namely for the empirical observation that longshots tend to be over-valued and that favorites tend to be under-valued (Ali 1977). More explicitly, consider a horse race with only two horses, and call the first horse the favorite (resp. longshot) if the mean beliefs that this horse wins are such that $\overline{p} \geq 1/2$ (resp. $\overline{p} \leq 1/2$). As we will see, the necessary and sufficient condition so that this horse is under-valued, i.e. $\pi^* \leq \overline{p}$, critically depends on whether it is a favorite or a longshot and on the risk aversion of traders. This result is presented in the following proposition in which $A(w) = -u''(w)/u'(w)$ denotes Arrow-Pratt's coefficient of absolute risk aversion and $P(w) = -u'''(w)/u''(w)$ denotes the coefficient of absolute prudence (Kimball 1990).

Proposition 4 *The prediction market price is greater than the mean beliefs of traders, i.e.* $\pi^* \geq \overline{p}$*, for all symmetric beliefs* $\widetilde{p}$ *about the mean belief* $\overline{p}$ *if and only if* $(1/2 - \overline{p})(P(w) - 2A(w)) \geq 0$ *for all* w *and* u *that imply a unique equilibrium.*

Proof: Recall that, when the equilibrium is unique, $\pi^* \geq \overline{p}$ if and only if $E\alpha(\widetilde{p}, \overline{p}) \geq 0$. For symmetric distributions, this holds true if and only if for all $\overline{p}$ (hereafter denoted p) we have

$$g(\delta) = \alpha(p + \delta, p) + \alpha(p - \delta, p) \geq 0, \tag{12}$$

in which $\alpha(p+\delta,p)$ is the unique solution of (8) and $\alpha(p-\delta,p)$ is the unique solution of (9) for $\delta \in [0, \min\{p, 1-p\}]$. Observe that $g(0) = 0$ and $g'(0) = 0$. Moreover, we have $g''(0) = 2\alpha_{pp}(p,p)$. Then, taking $\alpha_{pp}(p,p)$ from (10), we can see that $g''(0) \geq 0$ is equivalent to $(1/2-p)(P(w) - 2A(w)) \geq 0$ for all w. This provides the necessity part of the proposition.

We now prove the sufficiency. From (9), condition (12) is equivalent to

$$(p-\delta)(1-p)u'(w-\alpha(p+\delta,p)(1-p)) - (1-p+\delta)pu'(w+\alpha(p+\delta,p)p) \geq 0. \tag{13}$$

Denoting $\phi(x) = 1/u'(x)$ and $\alpha = \alpha(p+\delta,p) \geq 0$, $\pi^* \geq p$ is therefore satisfied if

$$(p+\delta)(1-p)\phi(w-\alpha p) - (1-p-\delta)p\phi(w+\alpha(1-p)) = 0 \tag{14}$$

implies

$$(p-\delta)(1-p)\phi(w+\alpha p) - (1-p+\delta)p\phi(w-\alpha(1-p)) \geq 0. \tag{15}$$

We now introduce two random variables:

$$\tilde{x} = \begin{cases} w+\alpha p, & \frac{p-\delta}{2p} \\ w-\alpha p, & \frac{p+\delta}{2p} \end{cases}, \qquad \tilde{y} = \begin{cases} w+\alpha(1-p), & \frac{1-p-\delta}{2(1-p)} \\ w-\alpha(1-p), & \frac{1-p+\delta}{2(1-p)} \end{cases}.$$

Then it can be verified that $E\tilde{x} = E\tilde{y} = w - \alpha\delta$ and $\tilde{x}$ is a mean-preserving spread of $\tilde{y}$ if and only if $p \geq 1/2$. Note that $\phi''(x) \geq 0$ if and only if $P \leq 2A$. Therefore, when $p \geq 1/2$ and $P \leq 2A$, we have

$$E\phi(\tilde{x}) \geq E\phi(\tilde{y}), \tag{16}$$

which is equivalent to

$$\begin{aligned} &\frac{1}{2p}\Big[(p-\delta)\phi(w+\alpha p) + (p+\delta)\phi(w-\alpha p)\Big] \\ \geq &\frac{1}{2(1-p)}\Big[(1-p-\delta)\phi(w+\alpha(1-p)) + (1-p+\delta)\phi(w-\alpha(1-p))\Big]. \end{aligned}$$

This last inequality then leads to

$$\begin{aligned} &(1-p)(p-\delta)\phi(w+\alpha p) - p(1-p+\delta)\phi(w-\alpha(1-p)) \\ \geq & -\Big[(1-p)(p+\delta)\phi(w-\alpha p) - p(1-p-\delta)\phi(w+\alpha(1-p))\Big] = 0, \end{aligned}$$

where the last equality is given by (14). This shows that the condition (15) is satisfied. Hence $\pi^* \geq p$ when $p \geq 1/2$ and $P \leq 2A$. Moreover, when $p \leq 1/2$, $\tilde{y}$ is a mean-preserving spread of $\tilde{x}$, and $\phi''(x) \leq 0$ is equivalent to (16), leading to $\pi^* \geq p$. The case $\pi^* \leq p$ under $(1/2 - p)(P - 2A) \leq 0$ can be demonstrated in an analogous fashion. This concludes the proof. ■

The difference between the mean beliefs $\overline{p}$ and the equilibrium price π^* therefore depends on whether the mean beliefs is less than 1/2, and on whether the absolute prudence P is greater than twice the absolute risk aversion $2A$. The sign of $P - 2A$ is a familiar condition on utility functions in the comparative statics of risk (Gollier 2001). In particular, for CARA utility functions, we have $P = A(< 2A)$. For CRRA utility functions $u(x) = x^{1-\gamma}/(1-\gamma)$, $P < 2A$ is equivalent to the constant relative risk aversion $\gamma > 1$, while $P > 2A$ is equivalent to $\gamma < 1$, and $P = 2A$ is equivalent to $\gamma = 1$, which corresponds to the logarithmic utility function.

The result in Proposition 4 is illustrated in Figure 1. The horizontal axis represents the mean beliefs $\overline{p}$ and the vertical axis represents the equilibrium price π^*. The diagonal therefore represents the case $\overline{p} = \pi^*$, which holds everywhere if and only if $P = 2A$ (i.e., u is logarithmic). The result therefore shows that there is a favorite-longshot bias if and only if the utility function displays $P > 2A$, or equivalently $\gamma < 1$ for the CRRA utility functions. This implies that, for all symmetric belief distributions, there is a favorite-longshot bias if and only if the traders are less risk averse than the logarithmic utility function. When the traders are more risk averse, a reverse favorite-longshot bias occurs. The result also generalizes the numerical simulations presented in Wolfers and Zitzewitz (2006) for CRRA utility functions and a set of specific distributions of beliefs.

Observe that the result in Proposition 4 is consistent with Propositions 2 and 3. Indeed, this result shows that the prediction market is unbiased under two extreme and separate conditions on the utility functions and the distribution of beliefs: either as in Proposition 2 when the utility is logarithmic ($P = 2A$) or as in Proposition 3 when mean beliefs equal one half for symmetric belief distributions.

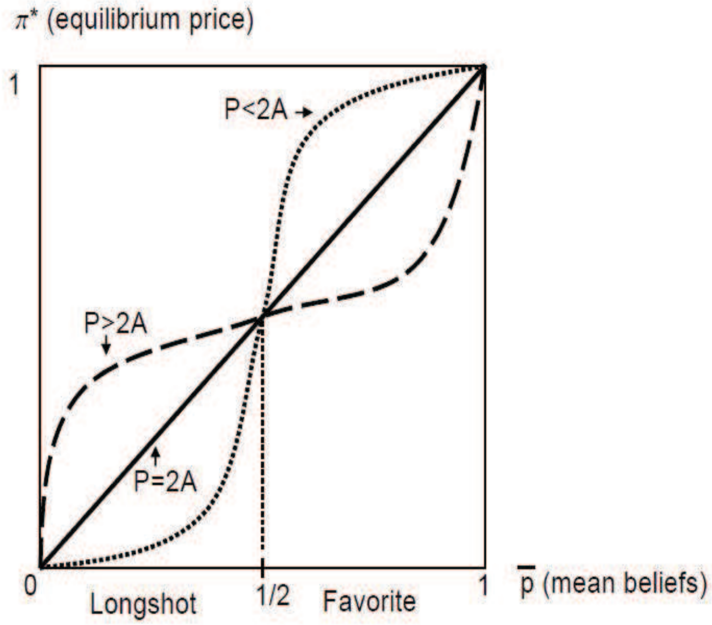

Figure 1: The equilibrium price π^* as a function of the mean beliefs $\bar{p}$. Under the symmetric beliefs, there is a "favorite-longshot bias" for the class of utility functions u satisfying $P(w) > 2A(w)$.

When traders are risk neutral, Ottaviani and Sorensen (2015, Proposition 2 and Corollary 1) show that underreaction to information leads to the favorite-longshot bias under the assumption of bounded wealth. They show that underreaction and hence the favorite-longshot bias becomes more pronounced for wider heterogeneous beliefs (measured by a mean preserving spread). When traders are risk averse, Ottaviani and Sorensen (2015, Proposition 6) show that a sufficient condition leading to a price underreaction to information is the strict DARA, which is equivalent to $P > A$. For CARA utility characterized by $P = A$, they show that there is no price underreaction. For all symmetric beliefs, the condition $P > 2A$ for the favorite-longshot bias in Proposition 4 is not only sufficient but also necessary. Our result is thus different from Ottaviani and Sorensen (2015, Proposition 6). In particular, when $A \leq P < 2A$ (including CARA utility), we obtain a reverse favorite-longshot bias.

One may wonder whether the condition $(1/2-\overline{p})(P(w)-2A(w)) \geq 0$ is also necessary and sufficient for all beliefs distributions $\widetilde{p}$, not only symmetric ones. To see this, note first that $\pi^* \geq \overline{p}$ is equivalent to $\sum_{i=1}^{N} \alpha(p_i, \overline{p}) \geq 0$, and since $\alpha(\overline{p}, \overline{p}) = 0$, by Jensen's inequality the necessary and sufficient condition for all $\widetilde{p}$ is simply given by $\alpha_{pp}(p, \overline{p}) \geq 0$ for all p and $\overline{p}$. The computation of $\alpha_{pp}(p, p)$ (see the proof of Proposition 2) shows that the condition $(1/2 - \overline{p})(P(w) - 2A(w)) \geq 0$ is indeed *necessary* for the favorite-longshot bias. However this condition is *not sufficient*, as the following example shows.

Example 4 (Nonsymmetric beliefs): Consider two groups of traders with $u(w) = \sqrt{w}$ (i.e., $P > 2A$) and heterogeneous beliefs $p_1 = 0.1$ and $p_2 = 0.9$. When the proportion of traders with beliefs $p_1 = 0.1$ is 75% then $\pi^* = 0.272 < \overline{p} = 0.3$ (i.e., the longshot is undervalued), and when the proportion of traders with beliefs $p_1 = 0.1$ is 25% then $\pi^* = 0.727 > \overline{p} = 0.7$ (i.e., the favorite is overvalued).

We believe that the general results and the selected numerical examples presented so far provide a fairly complete picture of the properties of equilibrium state prices of a prediction market in the two-state case. This is the case most often considered in the theoretical literature on prediction markets. We next extend our discussion to a prediction market with more than two states.

5 A discussion on the prediction market with many states

We have so far considered a prediction market with only two states. In this section we provide a discussion of a prediction market for any finite number S of states. We examine whether the previous results obtained for two states can be generalized to the case with more than two states. Consistent with previous notations, we denote by the vector $\mathbf{p}_i = (p_{i1}, ..., p_{is}, \cdots, p_{iS})$ the trader i's beliefs over states $s = 1, ..., S$ for $i = 1, 2, \cdots, N$, and by π_s the equilibrium price of state s. Similar to the two-state case, the prediction market is unbiased for state s when $\pi_s = \frac{1}{N}\sum_{i=1}^{N} p_{is}$.

As indicated in Section 3, a well-known result in the literature is that under a logarithmic utility function $u(w) = \log w$ prediction market is unbiased in the 2−state case (Gjerstad 2004, Wolfers and Zitzewitz 2006). It turns out that this sufficiency result can be generalized in the sense that all S state prices equal the mean of traders' beliefs for each state under $u(w) = \log w$. Since we have shown in Section 3 that $u(w) = \log w$ is also necessary for prediction market to be unbiased in the 2−state case, we can therefore state the following result.

Proposition 5 *(Generalization of Proposition 2) The prediction market is unbiased for all distributions of beliefs in the general $S-$state case if and only if $u(w) = \log w$.*

This result should not suggest, however, that the previous results can be directly generalized to the $S-$state case. Consider the previous observation that if all traders have the same beliefs about a particular state when $S = 2$ (and hence the same beliefs about the other state), then the prediction market is unbiased. This result is no longer valid in the general case. Indeed, we next show with the help of an example that, even if all traders have homogeneous beliefs about one state, the equilibrium price of that state could nevertheless be different from this homogeneous belief.

Example 5 (Prediction market bias under homogeneous beliefs): Let $N = 2$ and $S = 3$, and assume the following traders' beliefs: $\mathbf{p}_1 = (1 - 2p, p - \epsilon, p + \epsilon)$ and $\mathbf{p}_2 = (1 - 2p, p + \epsilon, p - \epsilon)$. Observe that the two traders have homogeneous beliefs $1 - 2p$ over state 1. Under CARA $u(w) = -e^{-rw}$ with $r > 0$, we have however $\pi_1 = \frac{1-2p}{1-2p+2\sqrt{p^2-\epsilon^2}} > 1 - 2p$ for any $\epsilon \neq 0$.

We now make another observation using the previous example. With the same numerical values for individual beliefs and utilities, we consider an alternative prediction market. Assume that there are only two Arrow-Debreu assets: an asset that pays \$1 if state 1 occurs, and another asset that pays \$1 if either state 2 or state 3 occurs. We have therefore a two-state prediction market. But since the traders' beliefs over the two states $(1-2p, 2p)$ are now homogeneous, we know that the prediction market is unbiased. Therefore, this simple observation shows that the equilibrium state price varies depending on the number of states on which it is possible to bet. In other words, this means that the "design" of the prediction market matters for equilibrium state prices.[6] Indeed, the intuition is that the design of the prediction market may affect trading opportunities under heterogeneous beliefs. Following this observation, one may ask: when is an equilibrium state price always independent from the design of the prediction market? It can be easily shown that this is the case when all traders believe that all other $S-1$ states are equally likely (see Proposition 6 below).

Example 5 also shows that the equilibrium price of one state depends on the distribution of beliefs about the other states (through the parameter ϵ). One may therefore ask: when does the equilibrium price of one state depend only on the beliefs about that state? Interestingly the answer to this question is exactly the same as the one to the question asked above about the prediction market design. Indeed, if the equilibrium price of one state in a $S-$state prediction market is always equal to the equilibrium price of that state in a two-state prediction market, this precisely means that the distribution of beliefs in all the other states is irrelevant for that equilibrium state price. We state this result in the following proposition.

Proposition 6 *For all u, the equilibrium price of one state, say π_1 for $s = 1$, in a $S-$state prediction market only depends on the beliefs about that state p_{i1} if and only if $p_{is} = p_i$ for all $s = 2, ..., S$, and for all $i = 1, ..., N$ (i.e. if and only if states 2 to S are judged as equally likely by all traders).*

Example 5 has shown that the prediction market may be biased despite homogeneous beliefs *in that state*. The next example shows the dual result that prediction markets

[6] To illustrate this interpretation, consider the horse race illustration. Example 5 and the last observation indicate that in a race with $S > 2$ horses, the equilibrium state price that one specific horse wins the race depends on whether it is possible to place bets separately on each of the other horses participating in the race.

may be unbiased despite heterogeneous beliefs. Specifically, Example 6 identifies a case with symmetric beliefs where the prediction market is unbiased for *any* CRRA utility functions.

Example 6 (Unbiased prediction market under heterogeneous beliefs): Let $N = 2$ and $S = 3$, and assume the following traders' beliefs: $\mathbf{p}_1 = (1/2 + 1/4, 1/3 - 1/6, 1/6 - 1/12)$ and $\mathbf{p}_2 = (1/2 - 1/4, 1/3 + 1/6, 1/6 + 1/12)$. Then under $u(w) = w^{1-\gamma}/(1-\gamma)$ with $\gamma > 0$, prediction market is unbiased for all states, i.e., $\pi_1 = 1/2, \pi_2 = 1/3$ and $\pi_3 = 1/6$.

Proposition 4 implies, in the two-state case and under CRRA utility functions, there is underpricing if and only if $\gamma > 1$, and thus that the prediction market is unbiased only in the knife-edge logarithmic case, i.e. $\gamma = 1$. In contrast, Example 6 illustrates that the equilibrium state price can be unbiased even when $P \neq 2A$. This further illustrates that the prediction market prices in the $S-$state markets can be very different from the 2-state prediction market prices.

6 Conclusion

In the last decades, academics and private-sector operators have increasingly used financial prediction markets with the primary objective to better predict future uncertain events. This paper has derived generic theoretical conditions to ensure the equilibrium state prices in such markets are equal to the mean beliefs of traders about that state. We have shown that these conditions are very stringent, even in a setting with homogeneous traders' risk preferences and wealth. We have also provided a set of additional conditions that are informative about how prediction market prices deviate from the mean beliefs of traders. In particular, we have identified an exact condition on risk aversion such that the favorite-longshot bias (Ali 1977) always holds for any symmetric belief distribution, and an exact condition on beliefs such that the equilibrium price of one state only depends on the heterogeneity of beliefs about that state.

A central assumption in our model is that traders do not update their subjective probabilities. With that assumption, we have followed the common setting adopted in the early economic papers on prediction markets (Gjerstad 2004, Manski 2006, Wolfers and Zitzewitz 2006). In a set of important papers, Ottaviani and Sorensen (2009, 2010, 2015) provide an alternative informational explanation to the prediction market favorite-longshot bias without relaxing standard Bayesian updating. For sure, in many environments, investors do update their beliefs based on new information. On the other hand, there is a lot of evidence that people are not Bayesian. Typically, the extensive research on the "confirmation bias" in psychology shows that people tend to search for, interpret, or recall information in a way that confirms their prior beliefs. This observation suggests that the prediction market prices may depend on the specific prediction market under consideration, and in particular on the preferences and rationality of participants. Ultimately, understanding the prediction market bias is an empirical question. Since our paper characterizes theoretically some simple relationships between the prediction market prices and the traders' beliefs and risk preferences, we believe that it can provide a good basis for empirical testing.

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du risque/SCOR.

Appendix: Proof of Results in Section 5

In this appendix, we setup an S-state prediction market, derive the equilibrium state prices, and provide the proofs of the Propositions and more details on the examples presented in Section 5.

A.1 The S-state model

Consider a prediction market with N traders, indexed by $i = 1, \cdots, N$, and S states, indexed by $s = 1, \cdots, S$. Traders have the same utility function $u(\cdot)$, however they have heterogeneous beliefs in the probability distribution over the states of nature, denoted by $\mathbf{p}_i = (p_{i1}, p_{i2}, \cdots, p_{iS})$ for trader i. Let π_s be the price of the Arrow-Debreu asset s that delivers \$1 in state s and \$0 in other states for $s = 1, \cdots, S$. Trader i chooses a portfolio $\alpha_i = (\alpha_{i1}, \cdots, \alpha_{iS})$ of the Arrow-Debreu assets to maximize his expected utility of portfolio wealth based on his belief $\mathbf{p}_i$. This leads to the standard first-order condition (FOC):

$$p_{is} u_i'(w_{is}) = \lambda_i \pi_s, \tag{A.1}$$

where λ_i is the Lagrange multiplier,

$$w_{is} = w_o + \alpha_{is} - \sum_{j=1}^{S} \pi_j \alpha_{ij}, \qquad i = 1, \cdots, N; \qquad s = 1, \cdots, S$$

is the portfolio wealth of trader i in state s, and w_o is the common initial wealth.

A.2 The equilibrium state prices

Based on the setup in subsection A.1, the equilibrium state prices $\{\pi_s\}$ are determined by the market clearing condition

$$\sum_{i=1}^{N} \alpha_{is} = 0, \qquad s = 1, \cdots, S.$$

To derive the equilibrium state prices, we consider three types of utility functions and the results are summarized in three Lemmas.

Lemma 3. For $u(x) = \log(x)$, the equilibrium state prices are given by

$$\pi_s = \frac{1}{N}\sum_{i=1}^{N} p_{is}, \qquad s = 1, \cdots, S, \tag{A.2}$$

that is, the state price is the mean probability belief of traders in the state.

Proof: With $u(x) = \log(x)$, the FOC (A.1) becomes

$$w_o + \alpha_{is} - \sum_{k=1}^{S} \pi_k \alpha_{ik} = \frac{1}{\lambda_i}\frac{p_{is}}{\pi_s}, \qquad i = 1, \cdots, N; s = 1, \cdots, S, \tag{A.3}$$

which leads to

$$\alpha_{is} = \alpha_{iS} + \frac{1}{\lambda_i}\left(\frac{p_{is}}{\pi_s} - \frac{p_{iS}}{\pi_S}\right), \qquad s = 1, \cdots, S.$$

Substituting the above expressions into (A.3) for $s = S$, we obtain that $\lambda_i = w_o$. This implies that the Lagrange multiplier λ_i is a constant in this case. Applying the market clearing condition to (A.3) then leads to the equilibrium state prices (A.2). ■

Lemma 4. For CARA utility $u(x) = -e^{-rx}/r$ with $r > 0$, the equilibrium state prices are given by

$$\pi_s = \frac{p_s^*}{\sum_{k=1}^{S} p_k^*} \qquad \text{with } p_s^* = \left(\prod_{i=1}^{N} p_{is}\right)^{1/N} \qquad \text{for } s = 1, \cdots, S. \tag{A.4}$$

Proof: With $u(x) = -e^{-rx}/r$, the FOC (A.1) becomes

$$\frac{p_{is}}{\pi_s} e^{-rw_{is}} = \lambda_i, \qquad i = 1, \cdots, N, \qquad s = 1, \cdots, S.$$

This leads to

$$\alpha_{is} - \sum_{k=1}^{S} \pi_k \alpha_{ik} = -w_o + \frac{1}{r}\left[\log\left(\frac{p_{is}}{\pi_s}\right) - \log(\lambda_i)\right], \qquad s = 1, \cdots, S. \tag{A.5}$$

Applying the market clearing conditions to (A.5), we obtain

$$rw_o + \frac{1}{N}\sum_{i=1}^{N} \log(\lambda_i) = \log\left(\frac{p_s^*}{\pi_s}\right), \qquad s = 1, \cdots, S, \tag{A.6}$$

where p_s^* is defined by $\log(p_s^*) = \frac{1}{N}\sum_{i=1}^N \log(p_{is})$ for $s = 1, \cdots, S$. Also, from (A.5),

$$\alpha_{is} = \alpha_{iS} + \frac{1}{r}\left[\log\left(\frac{p_{is}}{\pi_s}\right) - \log\left(\frac{p_{iS}}{\pi_S}\right)\right], \tag{A.7}$$

for $s = 1, \cdots, S$. Substituting (A.7) into (A.5) for $s = S$, we have

$$\sum_{s=1}^S \pi_s \log\left(\frac{p_{is}}{\pi_s}\right) = rw_o + \log(\lambda_i), \qquad i = 1, \cdots, N. \tag{A.8}$$

Aggregating (A.8) over i then leads to

$$rw_o + \frac{1}{N}\sum_{i=1}^N \log(\lambda_i) = \sum_{s=1}^S \pi_s \log\left(\frac{p_s^*}{\pi_s}\right). \tag{A.9}$$

Substituting (A.9) into (A.6), we then have

$$\log\left(\frac{p_s^*}{\pi_s}\right) = \sum_{k=1}^S \pi_k \log\left(\frac{p_k^*}{\pi_k}\right), \qquad s = 1, \cdots, S.$$

Therefore $p_s^*/\pi_s = \beta$ is a constant, independent of the state. Then (A.4) follows from $\sum_{s=1}^S p_s^* = \sum_{s=1}^S \pi_s\beta = \beta$. ■

Lemma 5. For CRRA utility $u(x) = x^{1-\gamma}/(1-\gamma)$ with $\gamma \neq 1$. the equilibrium state prices π_s satisfy

$$\pi_s^{1/\gamma} = \frac{1}{N}\sum_{i=1}^N \frac{p_{is}^{1/\gamma}}{\sum_{k=1}^S \pi_k \left(\frac{p_{ik}}{\pi_k}\right)^{1/\gamma}}, \qquad s = 1, \cdots, S. \tag{A.10}$$

Proof: With the CRRA utility function, the FOC (A.1) becomes $u'(w_{is}) = w_{is}^{-\gamma} = \lambda_i\pi_s/p_{is}$. Hence, with $g = -1/\gamma$, $w_{is} = \left(\frac{\lambda_i}{p_{is}}\pi_s\right)^g$. This, together with $w_{is} = w_o + \alpha_{is} - \sum_{k=1}^S \pi_k\alpha_{ik}$, leads to

$$\alpha_{is} = -w_o + \sum_{k=1}^S \pi_k\alpha_{ik} + \left(\frac{\lambda_i}{p_{is}}\pi_s\right)^g. \tag{A.11}$$

Equation (A.11) implies that, for $s = 1, \cdots, S$,

$$\alpha_{is} = \alpha_{iS} + \left[\left(\frac{\lambda_i}{p_{is}}\pi_s\right)^g - \left(\frac{\lambda_i}{p_{iS}}\pi_S\right)^g\right]. \tag{A.12}$$

Substituting (A.12) into (A.11) for $s = S$ and using the market clear condition, we obtain

$$w_o = \sum_{s=1}^{S} \pi_s \left(\frac{\lambda_i}{p_{is}} \pi_s \right)^g, \qquad i = 1, \cdots, N. \tag{A.13}$$

Also, applying the market clearing condition to (A.11), we have

$$w_o = \frac{1}{N} \sum_{i=1}^{N} \left(\frac{\lambda_i}{p_{is}} \pi_s \right)^g, \qquad s = 1, \cdots, S. \tag{A.14}$$

Combining (A.13) with (A.14) leads to the state prices π_s in (A.10). ■

A.3 Proofs of Propositions 5 and 6

The proof of Proposition 5 follows easily from the equilibrium state prices (A.2) in Lemma 3 for $u(x) = \log(x)$. To prove Proposition 6, we first show that $p_{is} = p_i$ for $s = 2, ..., S$ and $i = 1, ..., N$ implies $\pi_{2,S} = \sum_{s=2}^{S} \pi_s$, where $\pi_{2,S}$ denote the state price of an Arrow-Debreu security that delivers \$1 if either state $j = 2, ..., S$ occurs and \$0 if state 1 occurs. If $p_{is} = p_i$ for $s = 2, ..., S$, we have from the FOC (A.1) that

$$\frac{u'(w_{ij})}{u'(w_{ik})} = \frac{\pi_j}{\pi_k}, \text{ for all } i = 1, ..., N \text{ and } j, k = 2, ..., S.$$

Assume then by contradiction that $\pi_j > \pi_k$ for some $j, k = 2, ..., S$ and $j \neq k$. This implies $w_{ij} < w_{ik}$, and hence $\alpha_{ij} < \alpha_{ik}$ for all $i = 1, ..., N$. The last inequality cannot hold at the equilibrium due to the market clearing condition. As a result we must have $\pi_j = \pi_k \equiv \pi$ for all $j = 2, ..., S$. It is immediate that each trader i must demand the same amount, say α_i, for $j, k = 2, ..., S$. The problem of each trader i is then to select α_{i1} and α_i to maximize

$$p_{i1} u(w + \alpha_{i1} - \alpha_{i1}\pi_1 - \alpha_i(S-1)\pi) + (1 - p_{i1}) u(w + \alpha_i - \alpha_{i1}\pi_1 - \alpha_i(S-1)\pi).$$

This is equivalent to a binary-prediction market in which $(S-1)\pi$ denotes the equilibrium price of an Arrow-Debreu security that delivers \$1 if either state $j = 2, ..., S$ occurs. Therefore we have $\sum_{j=2}^{S} \pi_j = (S-1)\pi = \pi_{2,S}$. This implies the S-state problem is equivalent to a reduced two-state problem by combining states 2 to S into one state. Thus the equilibrium price of state 1 only depends on the beliefs of the state.

We now show that if there exists an individual i who assigns different probabilities for two states $j, k = 2, ..., S$ then we may always have $\pi_{2,S} \neq \sum_{j=2}^{S} \pi_j$. Consider a simple example with 3 states and 2 traders, with the following structure of beliefs: $\mathbf{p}_1 = (1-2p_1, p_1, p_1)$ and $\mathbf{p}_2 = (1-2p_2, p_2+e, p_2-e)$. Namely trader 1 judges states 2 and 3 as equally likely, while trader 2 judges states 2 and 3 as equally likely if and only if $e = 0$. With CARA preferences, in the three-state prediction market we use the equilibrium state prices in (A.4) of Lemma 4 and obtain $\pi_i = p_i^*/(p_1^* + p_2^* + p_3^*)$ for $i = 1, 2, 3$ with $p_1^* = \sqrt{(1-2p_1)(1-2p_2)}$, $p_2^* = \sqrt{p_1(p_2+e)}$ and $p_3^* = \sqrt{p_1(p_2-e)}$. However, in the two-state prediction market where states 2 and 3 are combined into one state, the beliefs $\mathbf{p}_1$ and $\mathbf{p}_2$ become $\bar{\mathbf{p}}_1 = (1-2p_1, 2p_1)$ and $\bar{\mathbf{p}}_2 = (1-2p_2, 2p_2)$, respectively. Hence the corresponding state prices become (using obvious notations) $\bar{\pi}_1 = \bar{p}_i^*/(\bar{p}_1^* + \bar{p}_{2,3}^*)$ and $\bar{\pi}_{2,3} = \bar{p}_{2,3}^*/(\bar{p}_1^* + \bar{p}_{2,3}^*)$ with $\bar{p}_1^* = \sqrt{(1-2p_1)(1-2p_2)}$ and $\bar{p}_{2,3}^* = 2\sqrt{p_1 p_2}$. Therefore $\bar{\pi}_{2,3} = \pi_2 + \pi_3$ for all p_1 and p_2 if and only if $e = 0$. A similar example can be generated for any arbitrary number of states. This completes the proof of Proposition 6.

A.4 Proofs of the results in Examples 5 and 6

In Example 5 with CARA utility function, we apply the equilibrium state price (A.4) in Lemma 4 to traders' beliefs and obtain that $p_1^* = 1 - 2p$ and $p_2^* = p_3^* = \sqrt{p^2 - \epsilon^2}$, leading to the state price π_1 in the example.

To show the result in Example 6, we apply the equilibrium state prices (A.10) in Lemma 5 for CRRA utility function. With $g = -1/\gamma$, the state prices π_s for $s = 1, 2, 3$ in this case satisfy

$$2 = (\pi_1)^g \left[p_{11}^{-g}/\Delta_1 + p_{21}^{-g}/\Delta_2\right], \tag{A.15}$$

$$2 = (\pi_2)^g \left[p_{12}^{-g}/\Delta_1 + p_{22}^{-g}/\Delta_2\right], \tag{A.16}$$

$$2 = (\pi_3)^g \left[p_{13}^{-g}/\Delta_1 + p_{23}^{-g}/\Delta_2\right], \tag{A.17}$$

where

$$\begin{aligned}\Delta_1 &= \pi_1(\pi_1/p_{11})^g + \pi_2(\pi_2/p_{12})^g + \pi_3(\pi_3/p_{13})^g,\\ \Delta_2 &= \pi_1(\pi_1/p_{21})^g + \pi_2(\pi_2/p_{22})^g + \pi_3(\pi_3/p_{23})^g.\end{aligned}$$

With the specified heterogeneous probabilities, $\Delta_1 = (12)^g\delta_1$ and $\Delta_2 = (12)^g\delta_2$, where

$$\begin{aligned}\delta_1 &= \pi_1(\pi_1/9)^g + \pi_2(\pi_2/2)^g + \pi_3(\pi_3)^g,\\ \delta_2 &= \pi_1(\pi_1/3)^g + \pi_2(\pi_2/6)^g + \pi_3(\pi_3/3)^g.\end{aligned}$$

Correspondingly, equations (A.15)-(A.17) lead to

$$2(\pi_1)^{-g} = 9^{-g}/\delta_1 + 3^{-g}/\delta_2, \quad \text{(A.18)}$$

$$2(\pi_2)^{-g} = 2^{-g}/\delta_1 + 6^{-g}/\delta_2, \quad \text{(A.19)}$$

$$2(\pi_3)^{-g} = 1/\delta_1 + 3^{-g}/\delta_2, \quad \text{(A.20)}$$

From (A.19) and (A.20), we obtain $\pi_3 = \pi_2/2$. Hence

$$\delta_1 = \pi_1(\pi_1/9)^g + (3/2)\pi_2(\pi_2/2)^g, \quad \text{(A.21)}$$

$$\delta_2 = \pi_1(\pi_1/3)^g + (3/2)\pi_2(\pi_2/6)^g. \quad \text{(A.22)}$$

Also, from (A.18) and (A.19),

$$3^g[\delta_1 + 3^g\delta_2]\pi_2^g = 2^g[3^g\delta_1 + \delta_2]\pi_1^g. \quad \text{(A.23)}$$

Substituting (A.21) and (A.22) into (A.23), we obtain

$$[3^g + 3^{-g}][\pi_1 - (3/2)\pi_2]\pi_1^g\pi_2^g = 3(2/3)^{1+g}\pi_2^{1+2g}[(\pi_1/\pi_2)^{1+2g} - (3/2)^{1+2g}],$$

leading to $\pi_1 = (3/2)\pi_2$.

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