Prediction Market Accuracy under Risk Aversion and Heterogeneous Beliefs

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Abstract

In this paper, we characterize the equilibrium price of a prediction market in which risk averse traders have heterogeneous beliefs in probabilities. We show that the prediction market is accurate, in the sense that the equilibrium price equals the mean beliefs of traders, if and only if the utility function of traders is logarithmic. We also provide a necessary and sufficient condition for the equilibrium price to be systematically below the mean beliefs for all symmetric belief distributions. In particular, when traders’ risk preferences are such that twice absolute risk aversion is less than absolute prudence, this condition provides a rationale for the favorite-longshot bias.

Keywords: Prediction market, heterogeneous beliefs, risk aversion, favorite-longshot bias.

JEL: D81, D84, G11, G12

*We thank Olivier Armantier, Kristoffer Glover, Christian Gollier, Charles Manski, François Salanié and Lei Shi, as well as seminar and conference participants at ESEM 2011, Paris-Dauphine and University of Technology Sydney (UTS) for comments. The first version of this paper was written while Nicolas Treich visited UTS in 2010. Financial support from the Quantitative Finance Research Center and the Paul Woolley Center at UTS is gratefully acknowledged. Tony He acknowledges funding for his subsequent visit in 2011 to Toulouse from the European Research Council under the European Community’s Seventh Framework Programme (FP7/2007-2013) Grant Agreement no. 230589 and financial support from the Australian Research Council (ARC) under Discovery Grant (DP130103210). Nicolas Treich also thanks the chair “Marché des risques et création de valeurs” Fondation du risque/SCOR.

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1 Introduction

Prediction markets are considered as an efficient tool to elicit people’s beliefs.\footnote{See for instance Hahn and Tetlock (2006). Prediction markets have been used to forecast market capitalization prior to an IPO (Berg et al. 2009), to test double auction in complex environments with few traders (Healy et al. 2010), and to evaluate the information aggregation and manipulation behavior (Jian and Sami 2012). See Goel et al. (2009) for a more contrasted view about the performance of prediction markets.} They have been used to predict the outcome of political elections, like with the Iowa electronic market. They are also used by private companies to elicit their employees’ beliefs about future sales or industry trends.

Technically, prediction markets are simple financial markets in which traders bet on the outcomes of uncertain events. Asset prices in prediction markets are typically interpreted as probabilities. Arrow et al. (2008) introduce prediction markets as follows: “Consider a contract that pays $1 if Candidate X wins the presidential election in 2008. If the market price of an X contract is currently 53 cents, an interpretation is that the market ‘believes’ X has a 53% chance of winning”.

In this paper, we consider a simple prediction market in which risk averse traders have heterogeneous beliefs in probabilities. We examine under which condition the prediction market is “accurate”, in the sense that the equilibrium state price is equal to the mean beliefs of traders.

Assuming that traders are risk neutral and have limited investment budgets, Manski (2006) presents a first formal analysis of this question. He shows that the equilibrium price of the prediction market can differ from the mean beliefs of traders. Wolfers and Zitzewitz (2006) consider a more standard model with risk averse traders. They show theoretically that the prediction market is accurate when the utility function of traders is logarithmic.\footnote{Moreover, Wolfers and Zitzewitz (2006) explore numerically how the equilibrium price is affected by belief heterogeneity for several utility functions and beliefs distributions. See Gjerstad (2004) for theoretical results under constant relative risk aversion (CRRA) utility functions, and Fountain and Harrison (2011) for numerical results allowing for wealth and beliefs heterogeneity. Page and Clemen (2013) and Gandhi and Serrano-Padial (2014) expand the model of Manski (2006) under risk neutrality and budget constraints.}

This paper derives a necessary and sufficient condition for prediction market accuracy for general utility functions and distributions of beliefs. Specifically, we show that the prediction market is accurate for all distributions of beliefs if and only if the utility function is logarithmic. Moreover, we present several examples in which the (joint) distributions of traders’ beliefs,
wealth and risk preferences lead to a systematic violation of prediction market accuracy. Yet, we provide indications about the direction of the bias. Most significantly, we exhibit a necessary and sufficient condition for the equilibrium price to be always below/above the mean beliefs depending on risk preferences and mean beliefs. More precisely, for all symmetric beliefs, there exists a favorite-longshot bias, meaning that the high-likelihood events are underpriced and the low-likelihood events are overpriced, if and only if the traders’ risk preferences are such that twice absolute risk aversion is less than absolute prudence.

2 The model

We consider a prediction market with only two states in which risk averse agents can buy or sell an asset paying $1 if an event occurs, and nothing otherwise. In this market, agents “agree to disagree” in the sense that they have heterogeneous prior beliefs. This means that the heterogeneity in beliefs comes from intrinsic differences in how agents interpret information, and not from asymmetric information. This assumption is common in the models of prediction markets cited above.

Let $u(\cdot)$ be the agent’s vNM utility function, which is strictly increasing, strictly concave and three times differentiable. Each agent maximizes over $\alpha$ the following expected utility

$$pu(w + \alpha(1 - \pi)) + (1 - p)u(w - \alpha\pi),$$  \hspace{1cm} (1)

in which $w$ his initial wealth, $p \in (0, 1)$ his subjective probability that the event occurs (i.e., his belief), $\alpha$ his asset demand and $\pi$ the price of this asset.\footnote{The individual asset demand $\alpha$ can be seen as the net asset demand of one asset in a model with two Arrow-Debreu assets. To see that, let $\alpha_s$ and $\pi_s$ denote respectively the demand for and the price of Arrow-Debreu assets in state $s = 1, 2$. The objective can then be written: $\max_{\alpha_1, \alpha_2} [pu(w + \alpha_1 - \pi_1\alpha_1 - \pi_2\alpha_2) + (1 - p)u(w + \alpha_2 - \pi_1\alpha_1 - \pi_2\alpha_2)]$. Denoting $\alpha = \alpha_1 - \alpha_2$ and observing that $\pi_1 + \pi_2 = 1$ by no arbitrage then leads (with $\pi = \pi_1$) to (1).}

The first order condition is given by

$$p(1 - \pi)u'(w + \alpha(p, \pi)(1 - \pi)) - (1 - p)\pi u'(w - \alpha(p, \pi)\pi) = 0,$$  \hspace{1cm} (2)

where $\alpha(p, \pi)$ is the solution to (2).
We now describe the equilibrium in this prediction market. Let \( \tilde{p} \) be the random variable representing the distribution of beliefs in the population of agents, and let \( \pi^* \) be the equilibrium price. The equilibrium condition is defined by

\[
E\alpha(\tilde{p}, \pi^*) = 0,
\]

in which \( E \) denotes the expectation operator with respect to \( \tilde{p} \).

Our main objective in the paper is to compare \( \pi^* \) to \( E\tilde{p} \). In particular, in Section 3 we will derive a condition so that there is prediction market accuracy defined by \( \pi^* = E\tilde{p} \). Since \( \alpha(p, p) = 0 \), notice immediately that, when \( \tilde{p} \) is degenerate and equal to \( p \) with probability 1, then \( \pi^* = p \). This is a trivial case always leading to prediction market accuracy.

An equilibrium always exists. Indeed, when \( \pi \) tends to 0 (resp. to 1) \( \alpha(p, \pi) \) becomes positive (resp. negative) for all \( p \), so its expectation over \( e \) also becomes positive (resp. negative). Therefore when \( \pi \) increases, the function \( E\alpha(\tilde{p}, \pi) \) goes from a positive to a negative region and thus must cross zero somewhere in between.

We now discuss the uniqueness of the equilibrium, that is whether \( E\alpha(\tilde{p}, \pi) \) only crosses the origin once. We know that \( \alpha(p, \pi) \) has this single crossing property at \( \pi = p \). But that does not guarantee that \( E\alpha(\tilde{p}, \pi) \) also has the single crossing property, as shown by the following example.

**Example 1** (Multiple equilibria): Consider agents with a quadratic utility function \( u(w) = -(1 - w)^2 \) and initial wealth \( w = 1/2 \). The optimal asset demand is equal to \( \alpha(p, \pi) = \frac{p - \pi}{2(p - 2\pi + p^2)} \). In a prediction market with only two agents with respective beliefs \( p_1 = 0.1 \) and \( p_2 = 0.9 \), the equilibrium condition is equivalent to \( 9 - 68\pi + 150\pi^2 - 100\pi^3 = 0 \). Solving for this equation, there are three equilibrium prices: \( \pi^* = (0.235, 0.5, 0.764) \).

Differentiating (2) with respect to \( p \), and rearranging, we obtain

\[
\alpha_p(p, \pi) = \frac{(1 - \pi)u'(w + \alpha(p, \pi)(1 - \pi)) + \pi u'(w - \alpha(p, \pi))}{-p(1 - \pi)^2u''(w + \alpha(p, \pi)(1 - \pi)) - (1 - p)\pi^2u''(w - \alpha(p, \pi))} > 0,
\]

that is, the asset demand increases with belief \( p \). Since \( \alpha(p, p) = 0 \), we conclude that \( \alpha(p, \pi) \geq 0 \) if and only if \( p \geq \pi \). Namely, the agent buys (resp. sells) the asset yielding $1 when the event occurs if and only if he assigns a probability for this event higher (resp. lower) than the asset price.
A sufficient condition for the uniqueness of the equilibrium however is \( \alpha_\pi(p, \pi) < 0 \). Indeed, this implies that the function \( E\alpha(\bar{p}, \pi) \) is strictly decreasing in \( \pi \), and therefore crosses zero at most once. Differentiating (2) with respect to \( \pi \), we have

\[
\alpha_\pi(p, \pi) = \frac{-pu'(w + \alpha(p, \pi)(1 - \pi)) - (1 - p)u'(w - \alpha(p, \pi)\pi)}{-p(1 - \pi)^2u''(w + \alpha(p, \pi)(1 - \pi)) - (1 - p)\pi^2u''(w - \alpha(p, \pi)\pi) - \alpha(p, \pi)\frac{p(1 - \pi)^2u''(w + \alpha(p, \pi)(1 - \pi)) - (1 - p)\pi^2u''(w - \alpha(p, \pi)\pi)}{-p(1 - \pi)^2u''(w + \alpha(p, \pi)(1 - \pi)) - (1 - p)\pi^2u''(w - \alpha(p, \pi)\pi)}}.
\]

The first term is strictly negative but the second term is of ambiguous sign. We now provide a sufficient condition for uniqueness by ensuring that the second term is also negative. (See the Appendix for Propositions’ proofs).

**Proposition 1** The equilibrium price \( \pi^* \) is unique if \( u \) has nonincreasing absolute risk aversion.

The intuition for this result is the following. When the price of an asset increases, there are two effects captured by the two terms of the right hand side of equation (5). First, there is a substitution effect that leads to a decrease in its demand. Second, there is a wealth effect that may increase its demand, and potentially dominate the first effect. Intuitively, as the terminal wealth distribution deteriorates, the investor’s attitude towards risk may change, and this wealth effect might prove sufficiently strong to increase the demand for the risky asset, as initially shown by Fishburn and Porter (1976). Under decreasing absolute risk aversion (DARA) however, the negative wealth effect leads the agent to be more risk averse, and therefore further decreases the demand for the risky asset. Under constant absolute risk aversion (CARA), there is no wealth effect, and only the first negative effect is at play. Finally, we note that Example 1 features multiple equilibria because the quadratic utility function has increasing absolute risk aversion.

### 3 A necessary and sufficient condition for prediction market accuracy

In this section, we first characterize prediction market accuracy by answering the following question: which utility functions lead to prediction market accuracy for all distributions of beliefs?
Assume first a logarithmic utility function, i.e. \( u(w) = \log w \) (which displays DARA).\(^4\) We can easily obtain a closed-form solution of the first order condition (2):

\[
\alpha(p, \pi) = w \frac{(p - \pi)}{\pi(1 - \pi)}.
\]

This condition implies that the equilibrium condition (3) can be written \( \pi^* = E\tilde{p} \). Thus, assuming a logarithmic utility function is sufficient for prediction market accuracy. This result is obtained in Gjerstad (2004) and Wolfers and Zitzewitz (2006). A natural question is whether the utility function must be logarithmic to guarantee prediction market accuracy or whether this is possible for other utility functions, i.e. whether \( u(w) = \log w \) is also a necessary condition. We show that this is indeed the case.

**Proposition 2** We have \( \pi^* = E\tilde{p} \) for all \( \tilde{p} \) if and only if \( u(w) = \log w \).

This result characterizes prediction market accuracy for all belief distributions. It is a knife edge case for the prediction bias examined in the next section. We complement this result with three remarks about its limitation in more general settings.

**Remark 1** (Wealth heterogeneity): The result of Proposition 2 cannot be generalized to non-identical wealth, as possible correlation between wealth and beliefs would invalidate the result. Indeed, let \( \tilde{w} \) be a random variable representing wealth heterogeneity. Assuming a logarithmic utility function, we obtain

\[
\pi^* = E\tilde{p} + \frac{1}{E\tilde{w}} \text{Cov}(\tilde{p}, \tilde{w}).
\]

Thus, there is no utility function ensuring prediction market accuracy when beliefs and wealth are potentially correlated. Observe that, despite this impossibility result, the direction of the bias can be inferred if the covariance between beliefs and wealth is known. The intuition for equilibrium condition (6) is that wealthier individuals invest more, and therefore have more influence on the equilibrium price. Thus, if wealth vary positively (resp. negatively) with beliefs, the equilibrium price is higher (resp. lower).

\(^4\)Since the vNM utility is defined up to an affine transformation, the statement \( u(w) = \log w \) should read in fact \( u(w) = a \log w + b \) for constants \( a > 0 \) and \( b \).
Remark 2 (Stakes): Suppose that each agent has a (positive or negative) stake $\Delta$ in the event he predicts, so that he now maximizes over $\alpha$ the following expected utility

$$pu(w + \Delta + \alpha(1 - \pi)) + (1 - p)u(w - \alpha\pi).$$

Then the result of Proposition 2 is not guaranteed either. Indeed for the logarithmic utility function the equilibrium condition is now defined by

$$\pi^* = \frac{w\hat{p}}{w + \Delta(1 - \hat{p})}.$$

When there is a positive (resp. negative) stake, the marginal utility decreases (resp. increases) if the event occurs. As a result, the agent wants to transfer wealth to the less favorable state, and uses the prediction market as a hedging scheme to do. The consequence is that the equilibrium price is biased.

Remark 3 (Quantities): Suppose that each agent can now bet on a random quantity payoff $\tilde{t}$. Formally, he chooses the amount $\alpha$ to maximize

$$E_{\tilde{t}}u(w + \alpha(\tilde{t} - \pi)),$$

where $E_{\tilde{t}}$ denotes the expectation operator over $\tilde{t}$. Note that we retrieve the previous prediction market model when $\tilde{t} = (1, 0; p)$, implying that prediction market accuracy holds for $u(w) = \log w$. Nevertheless prediction market accuracy fails when $\tilde{t}$ is not a binary, even for a logarithmic utility, as the following example shows. Suppose that $\tilde{t}$ can take three values, 1, 2 or 3. Consider two agents’ belief distributions over $\tilde{t}$ described respectively by $(1, 2, 3; 1/3, 1/3, 1/3)$ and $(1, 2, 3; 1/6, 1/6, 2/3)$. Thus, the first agent believes that the mean of $\tilde{t}$ is 2, while the second agent believes that it is 2.5. Solving for the equilibrium price under $u(w) = \log w$ and $w = 1$ we find $\pi^* = 2.227$, which is different from the mean beliefs of 2.25 across the two agents.

The previous Proposition provides the condition on the utility function so that there is market prediction accuracy for all belief distributions $\hat{p}$. We now study the dual problem: which conditions on $\hat{p}$ ensure prediction market accuracy for all $u$? We show that this depends on whether the probability distribution of beliefs is symmetric about one half.\(^5\)

\(^5\)Note that the two statements in Proposition 3 below should be mutually consistent. Therefore, the qualifier “for all $u$” means for all utility functions that lead to a unique equilibrium price (see Proposition 1 for a sufficient condition). A similar comment applies to Proposition 4.
Proposition 3 Assume that the equilibrium price $\pi^*$ is unique. Then, we have $\pi^* = E\tilde{p}$ for all $u$ if $\tilde{p}$ is symmetric about $1/2$.

The intuition for Proposition 3 is simple. When $\tilde{p}$ is symmetric about one half, the two states are formally indistinguishable, and therefore it cannot be that the price of an asset yielding one dollar in one state is different from that of an asset yielding one dollar in the other state, implying $\pi^* = 1/2$.

We note, however, that if heterogeneity in individual utility functions is introduced, prediction market accuracy may not hold anymore even when $\tilde{p}$ is symmetric about $1/2$. This is illustrated by the following example which considers heterogeneity over (constant absolute) risk aversion.

Example 2 (Heterogeneous CARA): Let $u_i(w) = -e^{-r_i w}$ in which $r_i > 0$ represents the CARA coefficient of agent $i = 1, 2$ with respective beliefs $p_1 = 0.1$ and $p_2 = 0.9$. Under positive correlation between beliefs and risk aversion $(r_1, r_2) = (1, 3)$, we have $\pi^* = 1/4 < 1/2 = E\tilde{p}$, while under negative correlation $(r_1, r_2) = (3, 1)$, we have $\pi^* = 3/4 > 1/2 = E\tilde{p}$.

We have characterized conditions for prediction market accuracy, for all $\tilde{p}$ in Proposition 2, and for all $u$ in Proposition 3. We can see that these conditions are rather stringent. We note, however, that one can relax these conditions in the sense that it is possible to find well-chosen pairs $(u, \tilde{p})$ also yielding prediction market accuracy. This is shown in the following example which uses a specific constant relative risk aversion (CRRA) utility function and a specific nonsymmetric distribution of beliefs.

Example 3 (Prediction market accuracy under nonsymmetric beliefs). Consider agents with utility function $u(w) = -1/w$. Two groups of agents participate in the prediction market: one group has beliefs $p_1 = p$, and the other group has beliefs $p_2 = 1 - p$. Denoting $a$ the proportion of agents in the first group, we have $E\tilde{p} = ap + (1 - a)(1 - p)$. One may then easily obtain that $E\alpha(\tilde{p}, \pi) = 0$ implies $\sqrt{\pi(1 - \pi)}\{ap + (1 - a)(1 - p) - \pi\} = 0$ leading to $\pi^* = E\tilde{p}$.

Examples 2 and 3 indicate that symmetry of beliefs about $1/2$ is neither necessary nor sufficient for prediction market accuracy in general.
4 A necessary and sufficient condition for the favorite-longshot bias

In the previous analysis, we have examined under which conditions the prediction market is accurate, i.e. $\pi^* = E\tilde{p}$. In this section, we derive a necessary and sufficient condition for a prediction bias, in the sense that $\pi^*$ is systematically above or below $E\tilde{p}$.

The analysis developed in this section may provide a rationale for the favorite-longshot bias, namely for the empirical observation that longshots tend to be over-valued and that favorites tend to be under-valued (Ali 1977, Thaler and Ziemba 1988, Gandhi and Serrano-Padial 2014). To understand our interpretation of the bias, consider a horse race with only two horses. Call the first horse the favorite (resp. longshot) if the mean beliefs that this horse wins is such that $E\tilde{p} \geq 1/2$ (resp. $E\tilde{p} \leq 1/2$). We show that the necessary and sufficient condition ensuring that a horse is under-valued, i.e. $\pi^* \leq E\tilde{p}$, depends in a systematic way on whether this horse is a favorite or a longshot, and on agents’ risk preferences. This result is presented in the following Proposition 4 in which $A(w) = -u''(w)/u'(w)$ denotes Arrow-Pratt’s absolute risk aversion, and $P(w) = -u''(w)/u''(w)$ denotes absolute prudence (Kimball 1990).

**Proposition 4** Assume that the equilibrium price $\pi^*$ is unique. Then, we have $\pi^* \geq E\tilde{p}$ for all symmetric $\tilde{p}$ if and only if $(1/2 - E\tilde{p})(P(w) - 2A(w)) \geq 0$ for all $w$.

The difference between the mean belief and the equilibrium price therefore depends on whether the mean belief is less than 1/2, and on whether absolute prudence is greater than twice absolute risk aversion. The sign of $P - 2A$ is a familiar condition on utility functions (Gollier 2001). Under CARA, $P = A (< 2A)$. Under CRRA, i.e. $u(x) = x^{1-\gamma}/(1 - \gamma)$ with $\gamma > 0$, $P < 2A$ is equivalent to $\gamma > 1$, while $P > 2A$ is equivalent to $\gamma < 1$, and $P = 2A$ is equivalent to $\gamma = 1$, which corresponds to the logarithmic utility function. Notice that DARA is equivalent to $P$ larger than $A$.

**INSERT FIGURE 1 ABOUT HERE**

The result in Proposition 4 is illustrated in Figure 1. The horizontal axis represents the mean belief and the vertical axis represents the equilibrium...
price. The diagonal therefore represents prediction market accuracy, which holds everywhere if and only if \( P = 2A \) (i.e., \( u \) is logarithmic). The result therefore shows that there is a favorite-longshot bias under \( P > 2A \), or equivalently \( \gamma < 1 \) under CRRA. Therefore, we provide a theoretical foundation for the favorite-longshot bias.

Observe that the result in Proposition 4 is consistent with Propositions 2 and 3. Indeed, there is prediction market accuracy under two extreme and separate conditions on the utility functions and the distribution of beliefs: either as in Proposition 2 when the utility is logarithmic (\( P = 2A \)) or as in Proposition 3 when mean beliefs equal \( 1/2 \) for symmetric belief distributions.

One may wonder whether the condition \((1 - 2)\exp(\alpha(\beta, \beta)) - 2A(w) \geq 0\) is also necessary and sufficient for all distributions \( \beta \), not only symmetric ones. To see this, note first that \( \pi^* \geq E\tilde{\beta} \) is equivalent to \( E\alpha(\tilde{\beta}, \beta) \geq 0 \), and since \( \alpha(\beta, \beta) = 0 \), by Jensen inequality the necessary and sufficient condition for all \( \tilde{\beta} \) is simply given by \( \alpha_{pp}(p, \bar{p}) \geq 0 \) for all \( p \) and \( \bar{p} \). The computation of \( \alpha_{pp}(p, \bar{p}) \) (in (A.1) of the Appendix) shows that the condition \((1/2 - E\tilde{\beta})(P(w) - 2A(w)) \geq 0\) is indeed necessary for the favorite-longshot bias. However this condition is not sufficient, as the following example shows.

**Example 4** (Failure of sufficiency under nonsymmetric beliefs): Consider two groups of agents with \( u(w) = \sqrt{w} \) (i.e., \( P > 2A \)) and heterogeneous beliefs \( p_1 = 0.1 \) and \( p_2 = 0.9 \). When the proportion of agents with beliefs \( p_1 = 0.1 \) is 75% then \( \pi^* = 0.272 < E\tilde{\beta} = 0.3 \) (i.e., the longshot is undervalued), and when the proportion of agents with beliefs \( p_1 = 0.1 \) is 25% then \( \pi^* = 0.727 > E\tilde{\beta} = 0.7 \) (i.e., the favorite is overvalued).

## 5 Conclusion

Under which conditions does a prediction market equilibrium price equal the mean of traders’ beliefs? Consistent with the early observation by Manski (2006), we have found that these conditions are stringent. However, we have provided a set of additional conditions that are informative about how prediction market prices vary with the belief distribution and the utility function of traders. In particular, we have derived a condition on risk aversion such that the favorite-longshot bias holds for all symmetric belief distributions.

A major assumption in our model is that traders do not update their beliefs once they see the price. With that assumption, we have followed
the common setting adopted in the early economic papers on prediction markets (Gjerstad 2004, Manski 2006, Wolfers and Zitzewitz 2006). As a promising alternative setting for prediction markets, Ottaviani and Sorensen (2010) recently studied a fully-revealing rational expectations equilibrium where traders make correct inferences from prices given common knowledge of the information structure and prior beliefs.

6 Appendix: Proof of Propositions 1-4

A.1 Proof of Proposition 1

We are done if we can show that the second term of the right hand side in (5) is negative. This is easy to show, and we only provide a sketch of the proof. Let \( \bar{x} = (1 - \pi, -\pi; p, 1 - p) \) denote a random variable \( \bar{x} \) which takes values of \( 1 - \pi \) and \( -\pi \) with probabilities \( \pi \) and \( 1 - \pi \), respectively. Then the first order condition (2) can be written more compactly \( \mathbb{E}[\bar{x}u'(w + \alpha \bar{x})] = 0 \). We are done if we can show that this last equality implies \(-\alpha \mathbb{E}[\bar{x}u''(w + \alpha \bar{x})] \leq 0\), that is, the second term of the right hand side in (5) is negative. This implication means that \(-u'\) is more risk averse than \(u\), which is equivalent to nonincreasing absolute risk aversion.

A.2 Proof of Proposition 2

We just need to prove the necessity. Namely, let \( \bar{p} = E\bar{p} \), we must show that \( E\alpha(\bar{p}, \pi) = 0 \) for all \( \bar{p} \) implies \( u(w) = \log w \). We are done if we can show that this implication holds for a specific class of probability distribution in \( \bar{p} \). We consider the class of "small" risks, in the sense that \( \bar{p} \) is close enough to \( \pi \) using the following second-order approximation: \( E\alpha(\bar{p}, \pi) = \alpha(\bar{p}, \pi) + 0.5E(\bar{p} - \bar{p})^2\alpha_{pp}(\bar{p}, \pi) \). The necessary condition \( E\alpha(\bar{p}, \bar{p}) = 0 \) then implies \( \alpha(\bar{p}, \pi) + 0.5E(\bar{p} - \bar{p})^2\alpha_{pp}(\bar{p}, \pi) = 0 \). Since for all \( p \) we have \( \alpha(p, p) = 0 \), the necessary condition becomes \( \alpha_{pp}(p, p) = 0 \). From (2), we obtain

\[
0 = 2\alpha_p(p, \pi)\{(1 - \pi)^2u''(w + \alpha(p, \pi)(1 - \pi)) - \pi^2u''(w - \alpha(p, \pi)\pi)\} + \alpha_{pp}(p, \pi)\{p(1 - \pi)^2u''(w + \alpha(p, \pi)(1 - \pi)) + (1 - p)\pi^2u''(w - \alpha(p, \pi)\pi)\} + \alpha_p(p, \pi)^2\{p(1 - \pi)^3u'''(w + \alpha(p, \pi)(1 - \pi)) - (1 - p)\pi^3u'''(w - \alpha(p, \pi)\pi)\}.
\]
Taking $\pi = p$ in the last expression, we have $\alpha_p(p, p) = \frac{1}{p(1-p)} \times \frac{u'(w)}{u''(w)}$ from (4), then rearranging yields

$$\alpha_{pp}(p, p) = \frac{(1 - 2p)}{p^2(1 - p)^2} \left[ \frac{u'(w)}{u''(w)} \right]^2 \left[ \frac{u'''(w)}{u''(w)} - 2 \frac{u''(w)}{u'(w)} \right]. \quad (A.1)$$

Therefore a necessary condition is $\frac{u''(w)}{u'(w)} = 2 - \frac{u''(w)}{u'(w)}$. Integrating this differential equation gives $u(w) = \log w$.

A.3 Proof of Proposition 3

We want to show that if $\tilde{p}$ is symmetric about $1/2$ then $E\tilde{p} = \pi^*$ for all $u$. Observe from the first order condition (2) that $\alpha(p, \pi) = -\alpha(1 - p, 1 - \pi)$. This implies that the equilibrium condition can be written $E\alpha(\tilde{p}, \pi^*) = E\alpha(1 - \tilde{p}, 1 - \pi^*) = 0$. Observe then that $\tilde{p}$ symmetric about $1/2$ means that $\tilde{p}$ is distributed as $1 - \tilde{p}$. Consequently the equilibrium condition implies $E\alpha(\tilde{p}, \pi^*) = E\alpha(\tilde{p}, 1 - \pi^*)$. Since the equilibrium is assumed to be unique, this last condition implies $\pi^* = 1 - \pi^*$, that is $\pi^* = 1/2 = E\tilde{p}$. ■

A.4 Proof of Proposition 4

Recall that, when the equilibrium is unique, $\pi^* \geq \overline{p}$ if and only if $E\alpha(\overline{p}, \overline{p}) \geq 0$. For symmetric distributions, this holds true if and only if for all $\overline{p}$ (hereafter denoted $p$) we have

$$g(\delta) = \alpha(p + \delta, p) + \alpha(p - \delta, p) \geq 0, \quad (A.2)$$

in which $\alpha(p + \delta, p)$ is the unique solution of

$$(p + \delta)(1 - p)u'(w + \alpha(p + \delta, p)(1 - p)) - (1 - p - \delta)pu'(w - \alpha(p - \delta, p)p) = 0 \quad (A.3)$$

and $\alpha(p - \delta, p)$ is the unique solution of

$$(p - \delta)(1 - p)u'(w + \alpha(p - \delta, p)(1 - p)) - (1 - p + \delta)pu'(w - \alpha(p + \delta, p)p) = 0 \quad (A.4)$$

for $\delta \in [0, \min\{p, 1 - p\}]$.

Observe that $g(0) = 0$ and $g'(0) = 0$. Moreover, we have $g''(0) = 2\alpha_{pp}(p, p)$. Then, taking $\alpha_{pp}(p, p)$ from (A.1), we can see that $g''(0) \geq 0$
is equivalent to $(1/2-p)(P(w) - 2A(w)) \geq 0$ for all $w$. This provides the necessity part of the Proposition.

We now prove the sufficiency. From (A.4), (A.2) is equivalent to

$$(p - \delta)(1 - p)u'(w - \alpha(p + \delta, p)(1 - p)) - (1 - p + \delta)p\phi(w + \alpha(p + \delta, p)p) \geq 0.$$  

(A.5)

Denoting $\phi(x) = 1/u'(x)$ and $\alpha = \alpha(p + \delta, p) \geq 0$, $\pi^* \geq p$ is satisfied if

$$(p + \delta)(1 - p)\phi(w - \alpha p) - (1 - p - \delta)p\phi(w + \alpha(1 - p)) = 0$$  

(A.6)

implies

$$(p - \delta)(1 - p)\phi(w + \alpha p) - (1 - p + \delta)p\phi(w - \alpha(1 - p)) \geq 0.$$  

(A.7)

We now introduce two random variables:

$$\tilde{x} = \begin{cases} w + \alpha p, & \frac{p - \delta}{2p}, \\ w - \alpha p, & \frac{p + \delta}{2p}, \end{cases} \quad \tilde{y} = \begin{cases} w + \alpha(1 - p), & \frac{1 - p - \delta}{2(1 - p)}, \\ w - \alpha(1 - p), & \frac{1 - p + \delta}{2(1 - p)}. \end{cases}$$

Then it can be verified that $E\tilde{x} = E\tilde{y} = w - \alpha \delta$ and $\tilde{x}$ is a mean-preserving spread of $\tilde{y}$ if and only if $p \geq 1/2$. Note that $\phi''(x) \geq 0$ if and only if $P \leq 2A$. Therefore, when $p \geq 1/2$ and $P \leq 2A$, we have

$$E\phi(\tilde{x}) \geq E\phi(\tilde{y}),$$  

(A.8)

which is equivalent to

$$\frac{1}{2p} \left[ (p - \delta)\phi(w + \alpha p) + (p + \delta)\phi(w - \alpha p) \right] \geq \frac{1}{2(1 - p)} \left[ (1 - p - \delta)\phi(w + \alpha(1 - p)) + (1 - p + \delta)\phi(w - \alpha(1 - p)) \right].$$

This last inequality then leads to

$$(1 - p)(p - \delta)\phi(w + \alpha p) - p(1 - p + \delta)\phi(w - \alpha(1 - p))$$

$$\geq - \left[ (1 - p)(p + \delta)\phi(w - \alpha p) - p(1 - p - \delta)\phi(w + \alpha(1 - p)) \right] = 0,$$

where the last equality is given by (A.6). This shows that the condition (A.7) is satisfied. Hence $\pi^* \geq p$ when $p \geq 1/2$ and $P \leq 2A$. Moreover, when $p \leq 1/2$, $\tilde{y}$ is a mean-preserving spread of $\tilde{x}$, and $\phi''(x) \leq 0$ is equivalent to (A.8), leading to $\pi^* \geq p$. The case $\pi^* \leq p$ under $(1/2 - p)(P - 2A) \leq 0$ can be demonstrated in an analogous fashion. This concludes the proof.
This figure plots the equilibrium price $\pi^*$ as a function of mean beliefs $\bar{p}$. Under symmetric beliefs, there is a “favorite-longshot bias” for the class of utility functions $u$ for which absolute prudence $P(w)$ is greater than twice absolute risk aversion $A(w)$. 
References


