

« Economics of Risk and Uncertainty »

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Objectives

- Introduce main concepts of risk theory (risk aversion..)
- Manipulate basic techniques (comparative statics..)
- Focus on economic principles and intuitions (rather than on, e.g., axiomatics)
- Theory-oriented but empirical (and experimental) aspects won't be ignored
- Will present behavioral models

Organization

- 1:30 hour-class on Mondays at 18:30 and Tuesdays at 17:00
- Last-year class slides may be downloaded on my webpage
- This year slides will be marginally and progressively updated
- Some exercises during the class ; questions welcome!
- M2 Environment + option for M2 finance and Deeqa students: compromise in the difficulty (class / exam) and in the selection of applied topics
- Advise for M2 students: please catch up at the beginning, early parts of the class most difficult ones
- Work out with using the references (books, papers)
- Written exam in march, oral in september

Some Excellent Books

1. * Mas-Colell et al., 1995, *Microeconomic Theory*, Oxford UP, Chapter 6
2. * Kreps, 1988, *Notes on the Theory of Choice*.
3. ** Gollier, 2001, *The Economics of Risk and Time*, MIT Press
4. ** Hirshleifer and Riley, 1994, *The Analytics of Uncertainty and Information*, Cambridge UP
5. *** Kahneman, Slovic and Tversky, 1982, *Judgment under Uncertainty: Heuristics and Biases*, Cambridge UP.
6. *** Ingersoll, 1987, *Theory of Financial Decision-Making*, R & F Editors
7. *** Dixit and Pindyck, 1994, *Investment under Uncertainty*, Princeton UP
8. *** Viscusi, 1998, *Rational Risk Policy*, Princeton UP.

* Foundations (but difficult)

** Most useful for this class

*** Further explorations

Most Relevant Scientific Journals

- Top theory Journals: *Econometrica*, *Review of Economic Studies*, *American Economic Review*, *Journal of Economic Theory*..
- Top field Journal: *Journal of Risk and Uncertainty*, *Journal of Finance*, *JEEM*, *Journal of Public Economics*
- Other Approaches: *Management Science*, *Risk Analysis*, *Journal of Legal Studies*

Outline

1. Introduction (#1h30)
2. Basics on Expected Utility (#4h30)
3. The Standard Portfolio Model (#3h)
4. Consumption and Risk (#3h)
5. Physical Risks (#3h)
6. Option Values (#3h)
7. Experimental Evidence (#3h)
8. Uncertainty and Ambiguity (#3h)
9. Behavioral Economics (#3h)

2. Basics on Expected Utility (EU)

Outline

- The EU paradigm
- Risk-aversion
- Certainty equivalent, risk premium
- Comparative risk-aversion
- Some utility functions
- Stochastic dominance
- Index of Riskiness
- Diversification

3. The Standard Portfolio Model

Outline

- The model and its basic properties
- Impact of risk aversion
- The equity premium puzzle
- Insurance
- Uncertain Pollution

4. Consumption and Risk

Outline

- A simple two-period model
- The notion of « prudence »
- How large are precautionary savings?
- Savings and portfolio choices
- Recursive preferences

5. Physical Risks

Outline

- State-dependent utility
- A prevention model
- WTP/ WTA
- The value-of-statistical-life
- Human capital
- Life insurance

6. Option Values

Outline

- The concept of information
- Information value
- The « irreversibility effect »
- The precautionary principle
- Application to climate change

7. Experimental Evidence

Outline

- Tests of expected utility
- Probability triangle
- Preference reversal
- The debate about incentives
- Are subjects risk-averse?
- Field experiments

8. Uncertainty and Ambiguity

Outline

- Risk perceptions
- Probability elicitation, scoring rule
- Mistakes, heuristics and biases
- Hindsight bias, overconfidence
- Bounded rationality
- Ellsberg's paradox
- A theory of ambiguity

9. Behavioral Economics

Outline

- Rank-dependent expected utility
- Prospect theory, loss aversion
- Regret and disappointment
- Anxiety and optimal beliefs
- Emotional decision-making
- Reference-dependent preferences

1. Introduction

Why is the study of risk important?

Most (if not all) decisions involve the future, and so involve risk:

- Private decisions (education, occupation, health, savings, financial investments..)
- Public decisions (fluctuations reduction, resources management, energy, global risks..)

Why is the Study of Risk Important? (cont'd)

Huge risk-management sectors: insurance and finance

Strengthen your education in theory

Risk theory models and concepts are instrumental for the economics of information

Introduction (cont'd)

Does risk matter macroeconomically?

Yes,

- The management of risk is what best explains economic development (Bernstein, 1998)
- Environmental risks (e.g., climate change, damages estimates may reach 5% to 20% of GDP, Stern, 2007)

But,

- Macroeconomic risk rather small (Lucas, 1987): standard deviation of the growth rate only about 3%
- Governments can pool risks and are 'risk-neutral'
- Technically: Risk is a second-order effect

What is Risk?

- Many definitions
- Here: a risk will be a random variable; say (loosely) a probability distribution over outcomes

Distinction risk vs. uncertainty

Known vs. unknown probability distribution over outcomes

Two classical approaches to uncertainty:

- Bayesian updating, starting from one prior, there will be different (posterior) probability distributions
- Ambiguity: multi-prior probability distributions

A brief history of some influential works on risk

- 1654: Pascal, Fermat: 'discovery' of the theory of probabilities
- 1703: Jacob Bernouilli's law of large numbers
- 1730: De Moine's bell curve
- 1738: Dan Bernouilli's decreasing marginal utility
- 1750: Bayes' law
- 1875: Galton's regression to the mean
- 1921: Keynes' and Knight's books on uncertainty
- 1944: von Neumann and Morgenstern's seminal book
- 1952: Markowitz's portfolio approach
- 1954: Savage's foundations of statistics
- 1964: Pratt's coefficient of risk-aversion
- 1979: Tversky and Kahneman's prospect theory

Influential Research Field

Nobel-awarded economists for their contributions on risk-related issues:

- Allais (1988)
- Markowitz, Miller and Sharpe (1990)
- Merton and Sholes (1997)
- Kahneman and Smith (2002)

Contributors: Samuelson (1970), Arrow (1972), Friedman (1976), Simon (1978), Tobin (1981), Modigliani (1985), Stiglitz (2001), Schelling (2005)...

Expected Value

Why not taking the « expected value » after all?

- Any other criterion, if used repeatedly will almost certainly leave less well off at the end; but infinite repetition is rarely an option
- Would you bet \$100 on the outcome of the flip of a coin? \$100,000?
- People buy insurance (transform risk into its expected value and pay a premium for this)
- Risk premia on financial markets
- However, people gamble! (casino, PMU...)

Expected Value (cont'd)

Why not taking the « expected value » after all?

- St Petersburg's paradox (1713):

A fair coin will be tossed until a head appears; if the first head appears on the n th toss, then the payoff is 2^n ducats. How much should one pay to play this game?

-The paradox is that expected value is infinite:

$$\sum_{n=1}^{\infty} (2^n) \times (1/2)^n = 1 + 1 + \dots = \infty$$

- Dan Bernouilli (1738)'s idea is twofold, i) utility depends on wealth $u(w)$ and ii) diminishing marginal utility, e.g. $u(w) = \log(w)$:

$$\sum_{n=1}^{\infty} \log(2^n) \times (1/2)^n < \infty$$

Another Argument

Why not taking the « expected value » after all?

- Multiplicative risks
- Consider an investment offering a 50/50 chance of multiplying by 1.5 (a 50% return) or by 0.6 (a 40% loss) the amount invested
- It has a positive expected value but, if repeated, will almost surely lead to ruin (because $0.6 \times 1.5 < 1$)

And the Variance?

Why not adopting a « mean-variance » criterion?

Probably a good approximation in many cases (e.g., for fairly « small » risks)

A = (99.9%, -\$1; 0.1%, \$999),

B = (99.9%, \$1; 0.1%, -\$999)

Which one do you prefer?

Same mean, same variance.

A is positively skewed, B negative skewed.

Third moments (and others) may matter.

The theory that will be presented may potentially account for preferences over any moments of the distribution

Risk Heuristics

- Detection test of a disease (women's breast cancer)
- Breast cancer prevalence at 40 years old: 1%
- The test is imperfect:
 - 90% with a breast cancer are tested "positive"
 - 10% with no breast cancer are also tested "positive"
- What is the probability that a person tested positive has actually the disease?

Base-Rate Fallacy

- Answer: about #8%
 - Harvard Medical School students and staff: half people answer 90%
 - Ignorance of the a priori distribution, misapplication of the Bayes' law
 - Framing effects
- => “Irrationalities”, “Mistakes”...

Normative vs. Positive Approach

- Normative: How **should** people behave in face of risks? Which risk policies **should** the government implement?
- Positive: How **do** people and governments actually behave in face of risks?
- The two approaches are different (e.g., due to « mistakes »)
- Mostly concentrate on normative approach (based on axioms of « rationality »)
- Positive aspects won't be ignored: see experimental evidence and behavioral economics

Important overlooked topics

- No strategic aspects (individual decision-making only)
- (Almost) nothing on markets, and thus nothing on risk-sharing, on asset pricing..
- (Almost) nothing on sophisticated financial products (derivatives..)

2. Basics on Expected Utility (EU)

Outline

- The EU paradigm
- Risk-aversion
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- Stochastic dominance
- Index of riskiness
- Diversification

Choices and Consequences

Only consequences matter (the process does not count, framing irrelevant...)

Choices may affect consequences

Consequences will (most often) take the form of a summary variable (eg, wealth)

Lotteries

A (random variable or a) lottery X is described by a vector : $X = (p_1, x_1; p_2, x_2; \dots)$ with $\sum_i p_i = 1$.

A compound lottery is a lottery whose outcomes are lotteries

Only probabilities and outcomes are relevant (no matter they come from a simple or a compound lottery)

Remark: A lottery L can be exhaustively defined in terms of probabilities only

From Axioms to a Theory

- Which lotteries « should » individuals prefer?
- Introduce axioms, namely starting assumptions from which other statements are logically derived
- The statements here will form a theory of choice: the expected utility theory
- It is the dominant decision theory of choice in risk situations

Axioms of Preferences

Let $R \succeq$ denote a preference relation over lotteries

Axiom 1: Completeness

For any pair of lotteries (L^a, L^b) , either $L^a R \succeq L^b$ or $L^a R \preceq L^b$ (or both)

Axiom 2: Transitivity

$L^a R \succeq L^b$ and $L^b R \succeq L^c$ implies $L^a R \succeq L^c$

Axiom 3: Continuity

Let $L^a R \succeq L^b R \succeq L^c$, there exists a scalar $\alpha \in [0, 1]$ such that $\alpha L^a + (1 - \alpha)L^c R \succeq L^b$ and $\alpha L^a + (1 - \alpha)L^c R \preceq L^b$

Axioms of Preferences

Axiom 4: Independence

For all $\alpha \in [0, 1]$:

$L^a R \geq L^b$ is equivalent to $\alpha L^a + (1 - \alpha)L^c R \geq \alpha L^b + (1 - \alpha)L^c$

- Remark: $\alpha L^a + (1 - \alpha)L^c$ means having the lottery L^a with probability α and L^c with probability $1 - \alpha$
- Hence the independence axiom means that the preference ordering must be independent from any lottery mixing
- No parallel in consumer theory under certainty

The Independence Axiom

The historical focus of the dispute around EU

Crucial for linearity in probabilities (and so crucial for tractability!)

Normative appeal: Immune to « dutch books »

Ex: Suppose $L^a R \geq L^b$ and $L^a R \geq L^c$ but

$L^a R \leq 0.5 L^b + 0.5 L^c$ (contrary to the Ind. Axiom!)

(More on this, see Green, QJE, 1987)

The vNM Utility Function

Theorem: Suppose that the preference relation satisfies Axioms 1 to 4. Then it can be represented by a preference functional that is linear in probabilities. That is, there exists a scalar u_s associated to each outcome x_s such that $L^a R \succeq L^b$ if and only if:

$$\sum_s p_s^a u_s \geq \sum_s p_s^b u_s$$

Remarks

- Implication: Let two wealth distributions X^a and X^b . By the Theorem, there exists a vNM utility $u(\cdot)$ such that $X^a R \geq X^b$ iff $Eu(X^a) \geq Eu(X^b)$
- Ordering of lotteries is not affected by taking an increasing transformation of the expected utility Eu (but may be affected by taking an increasing non linear transformation of vNM utility u)
- Can be generalized to *subjective* probabilities by Savage (1954)

Illustration

Consider a lottery $X = (50\%, \$200; 50\%, -\$100)$

Do you accept it?

Evaluation of the lottery: There exists a vNM utility function $u(\cdot)$ such that:

- $u(w+X) = 0.5u(w+200) + 0.5u(w-100)$
- which must be compared to $u(w)$

Remark: Theoretical foundation for the Bernouilli (1738)'s intuition of a utility function

Risk Aversion

Definition: An agent is risk-averse (risk-lover) iff, for any level of initial wealth w , he never (always) prefers a random variable X to its mean EX

Jensen's Inequality: For any random variable X , $Ef(X) \leq f(EX)$ iff f is concave.

\Rightarrow An EU maximizer is risk-averse (risk-lover) iff his vNM utility is concave (convex)

\Rightarrow u linear « means » risk-neutrality

Risk Premium

Definition (Pure risk) : A pure risk is a zero-mean random variable

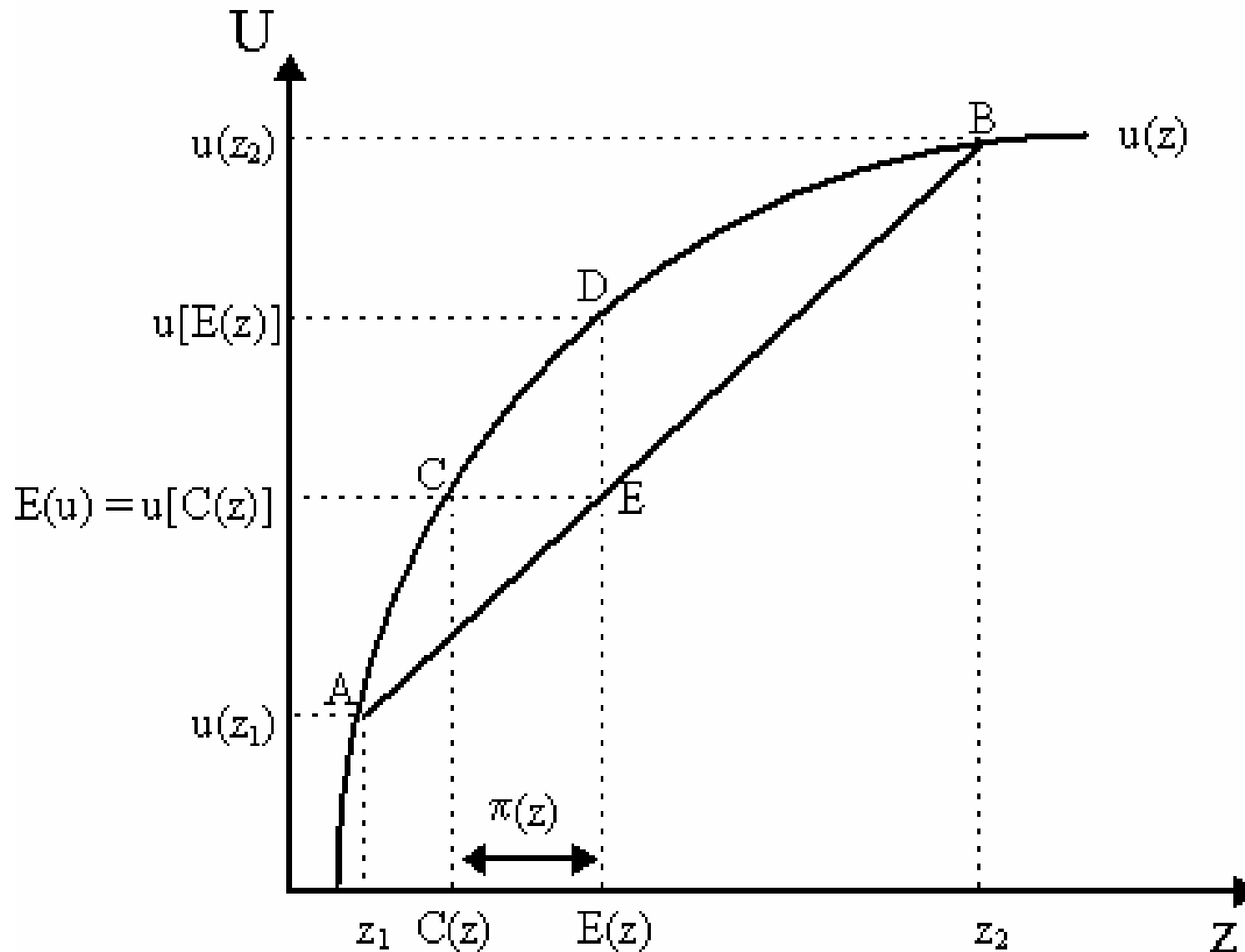
Definition (Risk Premium π) : Maximal amount of money that an individual is willing to pay to escape a pure risk

$$u(w - \pi(u, X)) = Eu(w + X) \text{ with } EX = 0$$

Certainty Equivalent C : Sure amount that makes the individual indifferent between accepting a lottery or not

$$w + C = u^{-1}(Eu(w + Y)) \text{ or } C = EY - \pi(u, Y)$$

Graph



Exercise 1: Lottery Comparisons

Consider 4 lotteries:

$X1 = (80\%, 4000; 20\%, 0)$, $X2 = (100\%, 3200)$,

$X3 = (20\%, 4000; 80\%, 0)$, $X4 = (25\%, 3200; 75\%, 0)$.

- i) Show that if an EU individual prefers $X1$ to $X2$ then he prefers $X3$ to $X4$.
- ii) Show that any risk-averse EU individual prefers $X2$ to $X1$ (and so $X4$ to $X3$).

Exercise 2: Insurance

An EU individual with initial wealth w may face a loss L with probability p . The loss L is random.

An insurer proposes to fully cover this risk provided the individual pays an insurance premium equal to $(1+\lambda)pEL$.

- i) Show that if $\lambda=0$, any risk-averse individual purchases insurance.
- ii) Does this hold if initial wealth is random? (and denoted $W=w+X$)

Exercise 2: Insurance (cont'd)

An EU individual with initial wealth w may face a loss L with probability p . The loss L is random.

An insurer proposes to fully cover this risk provided the individual pays an insurance premium equal to $(1+\lambda)pEL$.

- i) Show that if $\lambda=0$, any risk-averse individual purchases insurance.

$$(1-p)u(w)+pEu(w-L) \leq (1-p)u(w)+pu(w-EL) \leq u(w-pEL)$$

- ii) Does this hold if initial wealth is random? (and denoted $W=w+X$)

No correlation, yes (hint: let $v(w)=Eu(w+X)$ which is concave). Correlation, less clear.

Ex 3: The Firm's Problem

A firm is risk-neutral. After the firm learns the price p , she chooses production q to maximize profit:

$$pq - c(q)$$

with $c(q)$ convex.

Show that the firm prefers a risk on prices.

Ex 3: The Firm's Problem

A firm is risk-neutral. After the firm learns the price p , she chooses production q to maximize profit:

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with $c(q)$ convex.

Show that the firm prefers a risk on prices.

Let:

$$V(p) = \max_q pq - c(q)$$

$\Leftrightarrow V(p)$ convex in p (=risk-loving)

\Leftrightarrow Even under risk-neutrality, the firm prefers a risk on price!

More Risk Aversion?

- A risk neutral individual: u linear
- A risk averse individual: u concave
- How to characterize a « more risk averse » individual?

Risk-Aversion 'in the Small'

- $E u(w+kX) = u(w - \pi[k])$ with X a pure risk and u twice differentiable
- Compute the approximation of the risk premium: $\pi[k] \sim \pi[0] + k\pi'[0] + 0.5k^2 \pi''[0]$
- $= 0.5k^2 \pi''[0] = 0.5k^2 E X^2 (-u''(w)/u'(w))$

Risk-Aversion 'in the Large'?

- Yet we want a measure of the intensity of risk-aversion for **any** risk (not only small ones).
- Does not exist!
- But there is the following fundamental theorem about comparative risk aversion

Pratt (1964, Etca)'s Theorem

Theorem: The three following statements are equivalent. For all w and X ,

i) $-u''(w)/u'(w) \leq -v''(w)/v'(w)$

ii) For all X , $\pi[u,X] \leq \pi[v,X]$ (larger risk premium)

iii) There exists an increasing and concave transformation function T such that $v(w)=T(u(w))$

An individual with a vNM v is said to be more risk-averse than u iff $-u''(\cdot)/u'(\cdot) \leq -v''(\cdot)/v'(\cdot)$

Comparative Risk Aversion

- $-u''/u'$: often called the Arrow-Pratt coefficient of risk-aversion
- Comparison ceteris paribus, for all w and all X .
- There exist other coefficients of curvature:
 - Jones-Lee (1980, EJ): $u(.)'/(u^*-u(.))$ where u^* is the supremum of $u(.)$
 - Aumann and Kurz (1977, Etca): $u(.)'/u'(.)$
 - Mathematics' notion of curvature: $-u''(.)/(1+u'(.)^2)^{3/2}$
- The Arrow-Pratt coefficient is economically and mathematically intuitive, and fairly tractable

Risk Tolerance and DARA

Definition: Risk Tolerance is the reciprocal of risk aversion, that is $-u'/u''$

Definition: Relative Risk Aversion: $-zu''(z)/u'(z)$

Definition: A utility function $u(\cdot)$ displays Decreasing (resp. Increasing) Absolute Risk Aversion – denoted DARA (resp. IARA) – iff $-u''(w)/u'(w)$ is decreasing (resp. increasing) in w

DARA more intuitive

Exercises

- **Exercise 4.** Individual V is more risk-averse than individual U , and both have the same initial wealth. Show that individual V rejects all lotteries that individual U rejects.
- **Exercise 5.** Show that DARA implies i) u' convex, and ii) that u is less risk-averse than $-u'$
- **Exercise 6.**** Suppose at wealth w_0 an individual is indifferent between accepting or rejecting a lottery. Show that, under strict DARA, the lottery is accepted for all levels of wealth above w_0 . (hint: you may use Exercise 5)

Some Utility Functions

Linear: w

Quadratic: $-(a-w)^2$; $0 \leq w \leq a$

Exponential: $-\text{Exp}(-aw)$; $a \geq 0$ (or coined CARA)

Logarithmic: $\text{Log}(w)$

CRRA: $(w)^{1-\gamma}/(1-\gamma)$ with $\gamma \geq 0$

HARA: $a(b+w/\gamma)^{1-\gamma}$ with $b+w/\gamma \geq 0$

- **Exercise 7:** Compute the Arrow-Pratt's coefficient for these utility functions; Check whether DARA or not; check that HARA has linear risk tolerance

Are Governments' Risk Averse?

- No, governments should be « nearly » risk-neutral
- Standard argument: Governments can pool risks across the society (Arrow and Lind, 1970)
- Important implications for macroeconomics and risk policy-making

Intuition for the Government's Risk-Neutrality Argument

- Let an economy with n identical individuals, each with utility $u(w)$
- The government considers accepting a social risk Y , with $EY > 0$; individuals will each take an equal share of social risk: $V = Eu(w + Y/n)$
- Second-order Taylor approximation of welfare:
$$nV = nu(w) + EY u'(w) + (1/n) 0.5 EY^2 \times u''(w)$$
- Risk is a second order effect; moreover the risk term goes to 0 as n gets large

Implication

- Governments should accept any project with positive expected value (amounts to being risk neutral)
- Can really governments « pool » risks that way?
- Think of climate change for instance (correlated risks etc.)
- Related remark: the argument holds for financial risks (but may not hold for risks to life and health for instance)

Risk-aversion vs. Risk

- Pratt's theorem: keep the risk the same and compare different individuals (different utility functions)
- Help define the meaning of « more risk-averse » (Arrow-Pratt coefficient)
- From now, let's keep the individual the same, but change the level of risk
- Will help define the meaning of « more risk »

« More Risk » - An Example

Which one of the two lotteries do you prefer?

$X1 = (25\%, 300; 25\%, 100; 25\%, 0; 25\%, -200)$

$X2 = (50\%, 200; 50\%, -100),$

Show that any risk-averse EU maximizer
prefers $X2$ to $X1$

« More Risk » - Different Approaches

- Plot the density functions: A less-desirable lottery can be obtained by moving « weight » outside the support (preserving the mean)
- Adding noise: A less-desirable lottery is obtained by adding (contingent) pure risks
- Plot the cumulative functions: A less-desirable lottery has more « weight » on the left

Stochastic Dominance

Definition: Let F_1 and F_2 denote two cumulative distributions of X_1 and X_2 over support $[a,b]$. X_2 dominates X_1 in the sense of the second order stochastic dominance (denoted $X_2 \text{ SSD } X_1$) iff

$$D(t) = \int_a^t [F_2(s) - F_1(s)] ds \leq 0 \text{ for all } t$$

Definition (more common): A SSD that preserves the mean ($D(b)=0$) is called a « **decrease in risk** »

(Notice that: $E(X_i) = b - \int_a^b F_i(s) ds$)

Decrease in Risk

Theorem (Rothschild and Stiglitz, JET, 1970-71)

The three following conditions are equivalent:

- i) X_2 is a « decrease in risk » of X_1 (definition)
- ii) $E u(X_2) \geq E u(X_1)$ for all u concave
- iii) X_1 is obtained by adding (contingent) pure risks to X_2

Proof

(See RS 1970, Ingersoll 1987, p 137)

Sketch of the proof:

ii) equivalent to iii) : use Jensen's inequality

ii) equivalent to i) : start with ii) and make two successive integrations by parts

And the Variance?

- Is a decrease in risk equivalent to less variance?
- A decrease in risk does imply less variance. Take $u(\cdot)$ quadratic in $E u(X_2) \geq E u(X_1)$
- But less variance does **not** imply a decrease in risk. Example: Take $A=(50\%, 0.4; 50\%, 2.2)$ and $B=(1/9, 0.25; 7/9, 1; 1/9, 4)$. Then $E A = E B$ and A has a lower variance, but for instance $E \log A < E \log B$. Neither A or B stochastically dominates the other.

Diversification - Exercises

- **Exercise 8.** Should two commanders-in-chief of the Army take the same or a different plane?

(Assume the value of a commander-in-chief for the Army is L , that the Army is risk-averse and that any plane may crash with probability p)

- **Exercise 9**:** Let Z_1 and Z_2 be two i.i.d. random variables distributed as Z . Show that $X =_d 0.5(Z_1 + Z_2)$ is a decrease in risk compared to Z . Extend to N random variables.

First-Order Stochastic Dominance

Theorem: The three following conditions are equivalent:

- i) X_2 FSD X_1 (definition)
- ii) $E u(X_2) \geq E u(X_1)$ for all u increasing
- iii) $F_1(t) \geq F_2(t)$ for all t

Exercise 11: Proof

(hint: similar to the proof of the Rothschild and Stiglitz's theorem? Just one integration by part)

Meaning of the First-Order Stochastic Dominance (FSD)

- X_2 is « unambiguously » better than X_1
- For every outcome t , the probability of getting at least t is higher under F_2 than under F_1 .
- Remark: the cumulative functions must not cross

More on FSD/SSD

- Mas-Colell et al., 1995 – pages 194-99
- Gollier, 2001 – pages 39-48
- Hirshleifer and Riley, 1994 – pages 105-119
- Ingersoll, 1987 – pages 136-38

Main Limits of FSD/SSD

- Incomplete orderings (just a subset of random variables can be ordered)
- Not able to characterize the level of « riskiness » of a single random variable
- Needs an index of risk, that would parallel the Arrow-Pratt's index of risk aversion
- A good index should both increase with more dispersion (SSD), and a lower location (FSD)

Some Possible Indexes

- **Variance** – but does not depend on location
- **Entropy** ($\sum_k p_k \text{Log}_2 p_k$) – only depends on probabilities, does not depend on outcomes: absurd!
- (Inverse of the) **Sharpe ratio** (mean/standard deviation) – violates FSD
- **Value at risk at x%** (greatest possible loss ignoring losses with proba less than x%) – arbitrary parameter x, ignores the gain side of the lottery, violates FSD

Index of Riskiness**

- Aumann and Serano (2007, working paper)
- **Definition:** The index of riskiness $R(Z)$ of a lottery Z is defined by:
$$E \exp(-Z/R(Z)) = 1$$
- $R(Z)$ is the reciprocal of the risk aversion index of a CARA agent who is indifferent between accepting the lottery or not

Some Properties of the Index**

- Derived from axioms (Aumann and Serano, 2007)
- The riskiness of the lottery depends on the lottery only, and not (e.g.) on the wealth or utility function of the decision-maker
- Increases with more dispersion (SSD), and a lower location (FSD)
- $R(nZ) = nR(Z)$ for all positive numbers n
- If two lotteries have the same index of riskiness, any compound lottery has this index as well, and the sum of the lotteries also
- If Z has a normal distribution, then $R(Z) = 0.5 \text{Var}(Z) / EZ$

The Covariance Rule

This is a useful property (see, more generally, the diffidence theorem, Gollier, 2001)

The Covariance Rule:

Let $f(\cdot)$ be a weakly increasing function. Then $\text{COV}[f(X),g(X)] \leq 0$ for all X , if and only if $g(\cdot)$ is weakly decreasing.

Exercise 10.** Let $Z=kX$ with X a pure risk and $k>0$. Show that an increase in k induces an increase in risk. (Hint: one may use the covariance rule)

Aversion to Downside Risk**

Definition: An agent is *averse to a downside risk* if and only if he always prefers that a pure risk is contingent to a good outcome rather than contingent to a bad outcome

Theorem (Menezes et al., 1980, AER): An EU individual is averse to a downside risk if and only if his vNM utility $u(\cdot)$ is such that the marginal utility $u'(\cdot)$ is convex

Aversion to Downside Risk**

Sketch of the proof

Let X a pure risk and a random initial wealth W
distributed as $(1/N, w_1; 1/N, w_2; \dots; 1/N, w_N)$

Define $V_i = E_X u(w_i + X)/N + E_W u(W) - u(w_i)/N$
that is, the EU if the individual faces a pure risk
contingent on w_i

Write $N(V_i - V_j)$ and integrate

3. The Standard Portfolio Model

Outline

- The model and its basic properties
- Impact of risk aversion
- The equity premium puzzle
- Insurance
- Uncertain Pollution

Introductory Remark

The standard portfolio model is the paradigmatic model for analyzing risk-taking decisions

Idea: Examine the portion of the risk the decision-maker want to bear

Technically: The risk is multiplicative

This model is at the root of capital asset pricing models (CAPM), but variant versions include investment, insurance or pollution emissions models (as we will see)

The Standard Portfolio Model

An agent maximises a strictly increasing, concave and thrice differentiable vNM utility function u . Initial wealth is w_0

There exist only two assets: a riskless asset with rate of return r , and a risky asset with random rate of return R

The agent can invest an amount a (with $a \leq w_0$) in the risky asset, and $(w_0 - a)$ in the riskless asset

The Standard Portfolio Model

$$Eu((w_0 - a)(1 + r) + a(1 + R)) = Eu(w_0(1 + r) + a(R - r)),$$

that will be denoted $Eu(w + aZ)$ where $Z = R - r$ is the « net » return of the risky asset

Objective: examine the optimal risk-taking
decision: a^*

$$\text{FOC: } EZu'(w + a^*Z) = 0$$

$$\text{SOC: } EZ^2u''(w + a^*Z) < 0 \text{ under strict concavity of } u$$

Interiority Condition

$$\text{FOC: } EZu'(w+a^*Z)=0$$

Exercise 1: Show that $a^* > 0$ if and only if $EZ > 0$

Sketch of the proof: Let $g(a) = Eu(w+aZ)$;
Compute $g'(0)$

Remark: if Z is always positive (resp. negative) then a^* is equal to w_0 (resp. 0)

We will assume throughout that a^* is interior:
 $0 < a^* < w_0$

Small Risk 'Heuristics'

Let Z 'small' compared to w so that:

$$Eu(w+aZ) \approx u(w) + a(EZ)u'(w) + 0.5a^2(EZ^2)u''(w)$$

Maximizing over a the last expression leads to

$$a^* = (EZ/EZ^2)(-u'(w)/u''(w))$$

Increase in EZ , decrease in $\text{Var}(Z)$, increase in risk tolerance $-u'(w)/u''(w)$: intuitive effects

Exercise 2: Show that the above approximation is correct for u quadratic

Risk-aversion: Exercises

Exercise 3: Show that an increase in risk-aversion always leads to decrease a^*

Exercise 4: Show that if the utility function is CRRA, a^* is proportional to wealth w .

Exercise 5: Show that, under DARA, an increase in wealth w leads to increase a^*

Change in Risk

- Unlike the effect of more risk aversion, the effect of more risk in the sense of RS is ambiguous
- That is, it is not true in general that an increase (resp. decrease) in risk of Z leads to decrease (resp. increase) a^* for any risk-averse investor
- This is true, however, for some particular types of decrease in risk, for some probability distributions, and some utility functions (see, e.g., small risks, quadratic utility functions...)

Empirical Observations

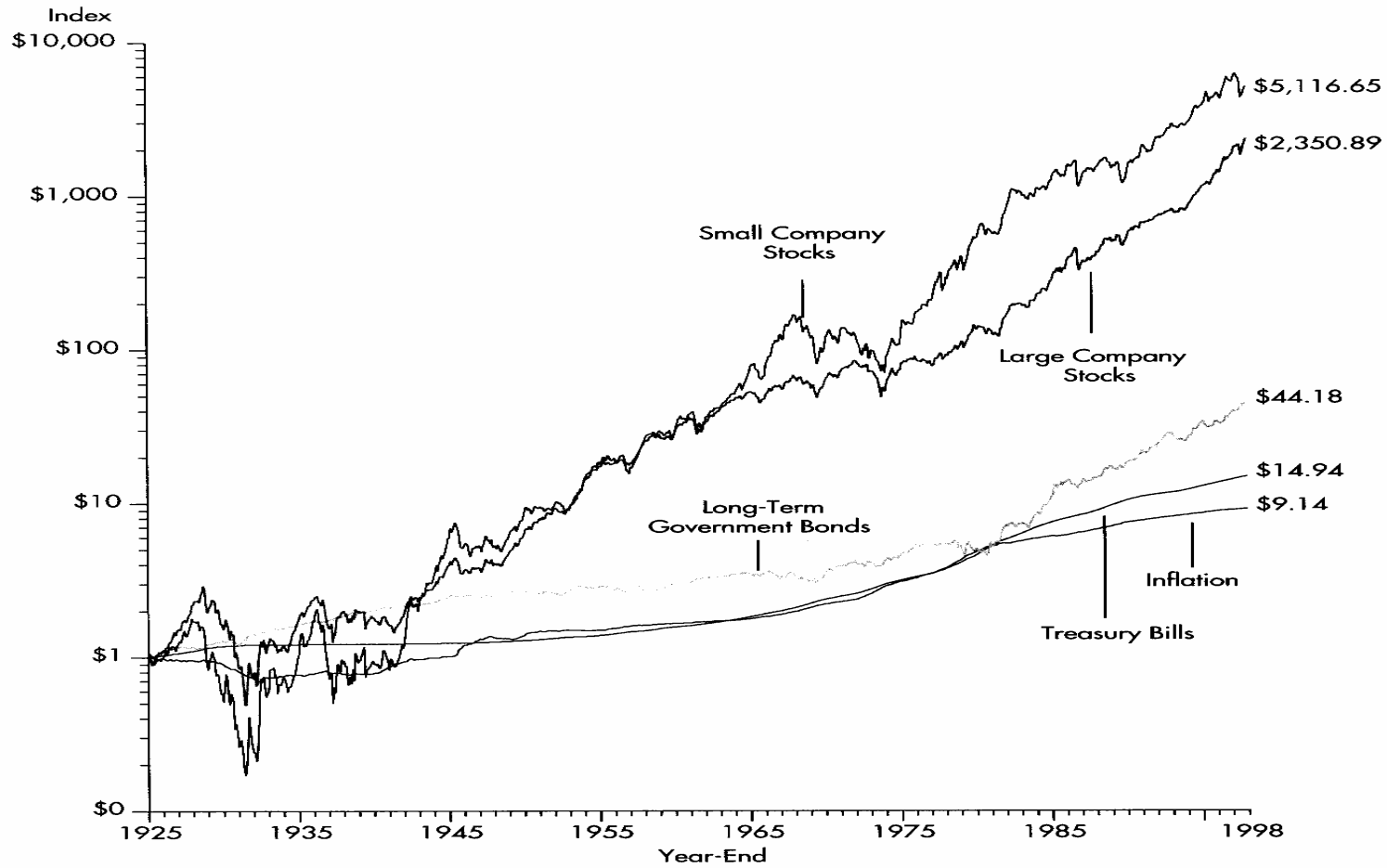
- Most people do not have any risky asset in their portfolio, only about 20% do have (« stock participation puzzle »)
- Rich people invest relatively more than poor people in risky assets. The equity share is not constant. See, e.g., Guiso et al. (AER, 1996)

Graph 2-1

**Wealth Indices of
Investments in the
U.S. Capital Markets**

Year-End 1925 = \$1.00

From 1925 to 1998



The Equity Premium Puzzle

- Need « extremely » high risk aversion to explain the equity premium ($\gamma > 40$)
- Mehra and Prescott (JME, 1985) and a voluminous subsequent literature
- Explanations?

Time Diversification

- Let $X=(50\%,\$200; 50\%,-\$100)$; Would you accept this bet? Would you accept $N=100$ bets in a row?
- Samuelson (1963, Scientia)'s Fallacy of the law of large numbers: if you refuse one such bet at every wealth level, you must refuse N bets.
- Intuition?

Time Diversification (cont'd)

- The variance (or the index of riskiness) increases with N
- Pratt and Zeckhauser's (1987, Etca) 'proper' utility functions: only requires the bet to be rejected at current weath (not at every wealth level as in Samuelson).
- But 'proper' is a quite stringent condition on the utility function

Insurance

An agent faces the risk of losing L (random) if an accident occurs: $Eu(w_0 - L)$

Insurance contract: get αL if accident, pay an insurance premium $P = (1 + \lambda)\alpha EL$ (where λ is an insurance loading factor, $0 \leq \alpha \leq 1$)

$$\begin{aligned} & Eu(w_0 - L + \alpha L - (1 + \lambda)\alpha EL) \\ &= Eu(w_0 - (1 + \lambda)\alpha EL + (1 - \alpha)((1 + \lambda)\alpha EL - L)) \end{aligned}$$

Insurance

$$Eu(w_0 - (1 + \lambda)EL + (1 - \alpha)((1 + \lambda)EL - L)) \\ = Eu(w + aZ)$$

where $w = w_0 - (1 + \lambda)EL$, $a = (1 - \alpha)$ and $Z = (1 + \lambda)EL - L$

- The choice a corresponds to the risk retention on the insurance market
- Observe that $EZ = \lambda EL$

Insurance

- Theoretical Predictions:
- Using the standard portfolio model (see before), we get:
 - 1) $a^* > 0$ iff $EZ > 0 \Leftrightarrow \alpha^* < 1$ iff $\lambda > 0$ (full insurance not optimal)
 - 2) DARA, a^* increases with wealth $w \Leftrightarrow \alpha^*$ decreases with wealth w

Empirical Puzzles in Insurance

- Inconsistent with theory:
 - 1) Full insurance policies are prevalent
 - 2) Insurance has often been found to be a normal good (increases with wealth)
- Explanations?

A Variant Version of the Model

Uncertain Pollution

Let:

e : (monetary-equivalent) benefit of pollution emissions

$d(e)$: damage function, $d'(\cdot) > 0$, $d''(\cdot) > 0$

$Xd(e)$: total uncertain damage from pollution emissions, where $X > 0$ is a random variable

Pollution Emissions

Thus the problem is : $\max_e Eu(w+e-Xd(e))$

Remark: Multiplicative Risk

$$\text{FOC: } E(1-Xd'(e))u'(w+e-Xd(e))=0$$

$$\text{SOC: } E(-Xd''(e))u'(w+e-Xd(e))+E(1-Xd'(e))^2u''(w+e-Xd(e))<0$$

Assume interior solution

Pollution Emissions

Exercise 7: Let the problem $\max_e Eu(w+e-Xd(e))$

- i) Characterize \bar{e} , the optimal emissions under certainty (X replaced by its mean). Show that it is identical to optimal emissions under risk neutrality

- ii) Show that uncertainty (or risk-aversion) reduces emissions compared to \bar{e}

Firm's Production

Exercise 8: Let the problem $\max_q \mathbb{E}u(Pq - c(q))$ with uncertain price $P > 0$ and with $c''(q) > 0$

- i) Characterize FOC, SOC
- ii) Show that optimal production under certainty, and under risk-neutrality are identical
- iii) Compare to $\arg \max_q \mathbb{E}u(Pq - c(q))$
- iv) Show, more generally, that an increase in risk-aversion reduces production under price uncertainty