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Catastrophe Aversion and Risk Equity in an Interdependent World

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Abstract. Catastrophe aversion and risk equity are important concepts both in risk management theory and practice. Keeney (1980) was the first to formally define these concepts. He demonstrated that the two concepts are always in conflict. Yet his result is based on the assumption that individual risks are independent. It has therefore limited relevance for real-world catastrophic events. We extend Keeney's result to dependent risks and derive the conditions under which more equity and more correlation between two risks imply a more catastrophic situation. We then generalize some of the results for multiple correlated risks.

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1. Introduction

The expected number of fatalities is perhaps the most common measure to assess and manage social risks. However, the expectation operation does not account for important dimensions of risk (Slovic et al. 1984). It neither reflects society's preferences to avoid large-scale accidents, nor does it capture concerns over inequalities in the distribution of risk across individuals (Bovens and Fleurbaey 2012). Alternative criteria for managing public risks have therefore emerged. These criteria aim at limiting the maximum probable loss or the maximum individual risk, reflecting society's anxiety to avoid a "bunching" of fatalities and its reluctance to accept risks that are unequally distributed across people.

Keeney (1980) was the first to formally define the concepts of risk equity and catastrophe aversion to capture both objectives. Risk equity is an ex ante concept corresponding to a preference for equalizing the probability of dying across agents. Catastrophe aversion, on the other hand, is an ex post concept corresponding to a preference for a mean-preserving concentration in the distribution of fatalities. Assuming independent risks, Keeney (1980) showed that the two concepts are always in conflict.[1] Whenever one increases risk equity, the distribution of fatalities becomes more catastrophic, and vice versa. This result is challenging as it highlights the conflict between two reasonable objectives of risk managers: limiting the risk burden to individuals and to society as a whole. It has received some attention in the operations research and management literature (e.g., Fishburn 1984; Keeney and Winkler 1985; Sarin 1985; Fishburn and Straffin 1989; Fishburn and Sarin 1991, 1994, 1997; Gajdos et al. 2010), and more recently in the economics and social choice literature (Bommier and Zuber 2008, Fleurbaey 2010, Bovens and Fleurbaey 2012, Adler et al. 2014, Rheinberger and Treich 2016).

In this paper, we examine an aspect of the problem that has so far been largely overlooked—the dependence structure of social risks. In today's world, interdependent risks are the rule rather than the exception. This observation is particularly true for potentially catastrophic risks such as hurricanes, terrorist attacks, climate change, or large-scale industrial accidents. In the remainder of the paper we gradually develop this argument. In Section 2 we show that the more correlated the risks faced by two agents are, the more *catastrophic* the corresponding distribution of fatalities is. We then derive in Section 3 the necessary and sufficient condition under which an equity-increasing risk transfer between two agents implies a more catastrophic distribution of risks. This condition pins down the effects that moving toward more risk equity has on both the marginal distributions of the two risks and on their correlation coefficient. We demonstrate that the condition holds whenever the risk transfer has a positive effect on the correlation between the two risks.

Extending the analysis to more than two agents is challenging because pairwise correlations provide an insufficient statistic to map out the dependence structure of multiple risks (Embrechts et al.

2002). Nonetheless, we generalize some of the results obtained for the two-agent case in Section 4 by imposing that, in an N-agent world, risk transfers between any two agents do not affect the dependence structure of the remaining $N-2$ agents, or by weakening the notion of more catastrophic. In Section 5, we discuss some implications for policy makers. Longer proofs appear in the appendix.

2. Catastrophic and Correlated Risks

2.1. Definitions and Notations

Consider a population of $i = 1, \ldots, N$ agents, each of whom faces an individual probability of dying $p_i \in [0, 1]$. The risk of death is modeled as a Bernoulli random variable $\tilde{x}_i$, which takes the value 1 (i.e., agent i dies) with probability p_i, and 0 otherwise. We are interested in the distribution of fatalities:

$$\tilde{d} := \sum_{i=1}^{N} \tilde{x}_i.$$

Following Adler et al. (2014), we define a more catastrophic distribution of fatalities based on the concept of second-order stochastic dominance (Rothschild and Stiglitz 1970).

Definition 1. Assume $\mathsf{E}[\tilde{d}] = \mathsf{E}[\tilde{d}']$. We say a distribution of fatalities $\tilde{d}$ is more catastrophic than another distribution $\tilde{d}'$ iff for any concave function f, $\mathsf{E}[f(\tilde{d})] \leq \mathsf{E}[f(\tilde{d}')]$.

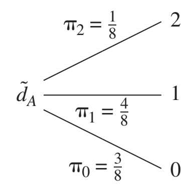
Assuming that $\tilde{d}$ is more catastrophic than $\tilde{d}'$ thus implies that $\tilde{d}$ is a mean-preserving spread of $\tilde{d}'$. Catastrophe avoidance as defined by Keeney (1980, theorem 2, p. 532) is a particular case of Definition 1, in which $\tilde{d}$ and $\tilde{d}'$ apply to binary risks with one safe outcome (i.e., an outcome that implies no fatality at all). Keeney's analysis assumes that the social planner applies the axioms of expected utility to evaluate the number of fatalities occurring in scenarios with different probabilities. Accordingly, catastrophe aversion can be formally defined as a concave (social) vNM utility function f.[2] Now, let $f(d) = -d^2$. It follows immediately that a more catastrophic distribution must have a greater variance, but not the other way around. This leads to our second definition.

Definition 2. We say a distribution of fatalities $\tilde{d}$ is more variable than another distribution $\tilde{d}'$ iff $\mathsf{var}(\tilde{d}) \geq \mathsf{var}(\tilde{d}')$.

Based on these two definitions, we will analyze the relationship between catastrophe aversion and risk equity. To begin with, we focus on two agents whose risks of death, $\tilde{x}_1$ and $\tilde{x}_2$, may be correlated. We introduce the probability space Ω, comprised of a finite (but potentially large) number of states S, to describe the dependence structure of the two risks. For now, let us assume that Ω has $S := 8$ equiprobable states $(\omega_1, \ldots, \omega_8)$ and consider two agents, 1 and 2, who face the probability of dying p_1 and p_2, respectively.[3]

Throughout the paper, we will make use of the following matrix notation to illustrate risky social situations:

$$\begin{array}{cc} & \begin{array}{cc} \tilde{x}_1 & \tilde{x}_2 \end{array} \\ \begin{array}{c} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \\ \omega_5 \\ \omega_6 \\ \omega_7 \\ \omega_8 \end{array} & \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \end{array}$$



(A)

Each of the $S := 8$ rows in this matrix corresponds to one possible state of the world. For $s = 1, \ldots, 8$, the ith value in row s can be interpreted as $\tilde{x}_i(\omega_s)$, which is the value taken by the random variable $\tilde{x}_i$ in state ω_s. In situation (A), there is only one state (ω_1) in which the two agents die simultaneously. Each state in situation (A) has the same probability $1/S = \frac{1}{8}$ to occur. $\tilde{x}_1(\omega)$ can take two values: 1 with probability $p_1 := \Pr(\tilde{x}_1 = 1) = \frac{1}{2}$ (corresponding to the occurrence of a state in which the first agent dies) and 0 with probability $1 - p_1 = \frac{1}{2}$ (corresponding to a state in which the first agent survives). Likewise, the probability that the second agent dies is $p_2 := \Pr(\tilde{x}_2 = 1) = \frac{1}{4}$. Note that $\tilde{x}_2$ does not depend on the realizations of $\tilde{x}_1$, and vice versa: $\Pr(\tilde{x}_2 = 1 \mid \tilde{x}_1 = 1) = \Pr(\tilde{x}_2 = 1)$ and $\Pr(\tilde{x}_1 = 1 \mid \tilde{x}_2 = 1) = \Pr(\tilde{x}_1 = 1)$. In other words, the two risks are independent.

Based on the matrix notation, the computation of the distribution of fatalities $\tilde{d}_A := \tilde{x}_1 + \tilde{x}_2$ is straightforward. We only need to sum the values in each row to find that the probabilities of observing zero, one, and two fatalities are equal to $\pi_0 := \Pr(\tilde{d} = 0) = \frac{3}{8}$, $\pi_1 := \Pr(\tilde{d} = 1) = \frac{4}{8}$, and $\pi_2 := \Pr(\tilde{d} = 2) = \frac{1}{8}$, respectively. The corresponding distribution of fatalities is represented by the probability tree next to the matrix.

2.2. Correlation and the Distribution of Fatalities

In a two-agent world, there is a fundamental relationship between the distribution of fatalities and the correlation of the risks (Meyer and Strulovici 2012). Proposition 1 captures the relationship assuming fixed marginal distributions (i.e., the parameters p_1 and p_2 are kept fixed). Thus, at this stage, there is no change in risk equity involved.

Proposition 1. *Under $N=2$, the four following statements are equivalent*:

(i) *the probability of simultaneous fatalities increases;*

(ii) *the correlation between the individual risks increases;*

(iii) *the distribution of fatalities is more catastrophic* (*Definition* 1);

(iv) *the distribution of fatalities is more variable* (*Definition* 2).

Proof. We first prove that the distribution becomes more catastrophic iff the probability of simultaneous fatalities, $\pi_2 := \Pr(\tilde{x}_1 = 1, \tilde{x}_2 = 1)$, increases. For $N=2$ agents, $\mathsf{E}[f(\tilde{d})] = \pi_0 f(0) + \pi_1 f(1) + \pi_2 f(2)$, where $\pi_i := \Pr(\tilde{d} = i)$ for $i = 0,1,2$. We know that $\mathsf{E}[\tilde{d}] = p_1 + p_2$, so that $\pi_1 + 2\pi_2 = p_1 + p_2$ (for $f(x) = x$) and $\pi_0 + \pi_1 + \pi_2 = 1$. Using these two equalities, we can express π_0 and π_1 as functions of π_2: $\mathsf{E}[f(\tilde{d})] = (1 - (p_1 + p_2 - 2\pi_2) - \pi_2) f(0) + (p_1 + p_2 - 2\pi_2) f(1) + \pi_2 f(2)$. This expression can be further simplified to

$$\mathsf{E}[f(\tilde{d})] = (1 - p_1 - p_2) f(0) + (p_1 + p_2) f(1) + \pi_2 (f(0) - 2f(1) + f(2)).$$

By Jensen's inequality we have $f(0) - 2f(1) + f(2) \le 0$ for all f concave. Therefore, $\mathsf{E}[f(\tilde{d})]$ decreases iff π_2 increases. Thus (iii) $\Leftrightarrow$ (i).

Next, we turn to the joint probability of two random Bernoulli variables. We have

$$\pi_2 = \Pr(\tilde{x}_1 = 1, \tilde{x}_2 = 1) = \mathsf{E}[\tilde{x}_1 \tilde{x}_2] = p_1 p_2 + \rho \sqrt{p_1(1-p_1)}\sqrt{p_2(1-p_2)}$$

by the definition of the correlation coefficient ρ between the two risks $\tilde{x}_1$ and $\tilde{x}_2$. Thus π_2 increases whenever ρ increases: (i) $\Leftrightarrow$ (ii). To conclude the proof, observe that

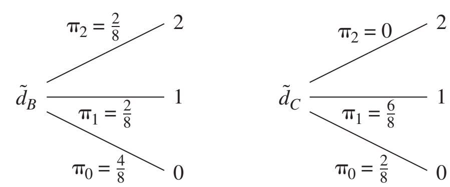
$$\mathrm{var}(\tilde{x}_1 + \tilde{x}_2) = p_1(1-p_1) + p_2(1-p_2) + 2\rho\sqrt{p_1(1-p_1)}\sqrt{p_2(1-p_2)},$$

which increases iff correlation ρ increases. Thus (ii) $\Leftrightarrow$ (iv). □

Let us provide an intuition for this result by modifying the introductory example (A). The distribution of fatalities can be more or less catastrophic depending only on the interaction between the two individual risks of death. In situation (B) below, we alter the dependence structure between $\tilde{x}_1$ and $\tilde{x}_2$ so that the occurrence of two simultaneous fatalities becomes most likely. By contrast, situation (C) illustrates a dependence structure in which two simultaneous fatalities are impossible:

$$\begin{array}{cc} \tilde{x}_1 & \tilde{x}_2 \end{array} \qquad\qquad \begin{array}{cc} \tilde{x}_1 & \tilde{x}_2 \end{array}$$
$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad\qquad \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

(B) (C)



Risky situation (B) implies the most catastrophic distribution of fatalities. This situation, in which the worst outcome (i.e., the joint death of two agents) becomes as likely as it can be given the marginals, is known as the *comonotonic* dependence structure. By contrast, risky situation (C) gives rise to the least catastrophic distribution of fatalities. This situation is also known as the *antimonotonic* dependence structure because outcomes are ordered in reverse order, thereby minimizing the probability to observe simultaneous fatalities. Indeed, $\tilde{d}_B$ is more catastrophic (and also more variable) than $\tilde{d}_C$.

The result in Proposition 1 is linked to Epstein and Tanny's (1980) concept of "generalized correlation." This concept characterizes the condition under which two random variables $\tilde{x}_1$ and $\tilde{x}_2$ are more correlated than two other random variables $\tilde{x}_1'$ and $\tilde{x}_2'$ (see also Tchen 1980 and Wright 1987). Epstein and Tanny show that this condition is formally equivalent to $\mathsf{E}[u(\tilde{x}_1, \tilde{x}_2)] \le \mathsf{E}[u(\tilde{x}_1', \tilde{x}_2')]$ whenever the cross-partial derivatives are nonpositive, i.e., $u_{12} \le 0$. Now, take $u(x_1, x_2) = f(x_1 + x_2)$ so that $u_{12} \le 0$ is equivalent to $f'' \le 0$. This proves that an increasing generalized correlation between $\tilde{x}_1$ and $\tilde{x}_2$ is necessary and sufficient for obtaining a more catastrophic distribution $\tilde{x}_1 + \tilde{x}_2$, with $\tilde{x}_1$ and $\tilde{x}_2$ being Bernoulli random variables.

From Proposition 1 we know that the distribution of fatalities becomes most (least) catastrophic when the correlation between the two risks is maximized (minimized). In the case of two agents, it is always possible to fully identify the least and most catastrophic distribution of fatalities. This result is summarized in the following lemma, and will turn out to be useful later. In particular, we notice that the range of correlation between two risks depends on their marginal probabilities.

Lemma 1. *The correlation ρ between two Bernoulli random variables $\tilde{x}_1$ and $\tilde{x}_2$ with $p_1 > p_2$ is bounded by*

- $\rho \in \left[-\frac{\sqrt{p_1 p_2}}{\sqrt{1-p_1}\sqrt{1-p_2}}, \frac{\sqrt{p_2}\sqrt{1-p_1}}{\sqrt{p_1}\sqrt{1-p_2}}\right]$, *if* $p_1+p_2 \leqslant 1$
- $\rho \in \left[-\frac{\sqrt{1-p_1}\sqrt{1-p_2}}{\sqrt{p_1 p_2}}, \frac{\sqrt{p_2}\sqrt{1-p_1}}{\sqrt{p_1}\sqrt{1-p_2}}\right]$, *if* $p_1+p_2 > 1$
- $\rho = 0$, *if* $p_1 = 1$ (*agent* 1 *is certain to die*) *or* $p_2 = 0$ (*agent* 2 *is certain to survive*).

Proof. The original proof of Lemma 1 is due to Meilijson and Nadas (1979) and Tchen (1980). Here, we only provide a sketch of the proof. We start with minimum correlation, which is attained when the Bernoulli variables are antimonotonic, i.e., when the number of simultaneous deaths is minimized. Specifically, we know that $\pi_2 = \mathsf{E}[\tilde{x}_1\tilde{x}_2] = \Pr(\tilde{x}_1 = 1, \tilde{x}_2 = 1) = \max(0, p_1 + p_2 - 1)$. There are two cases $p_1 + p_2 \leq 1$ and $p_1 + p_2 > 1$. In both cases, the minimum correlation is equal to

$$\frac{\max(0, p_1 + p_2 - 1) - p_1 p_2}{\sqrt{p_1(1-p_1)p_2(1-p_2)}}.$$

By considering these two cases separately and after some simplifications, one obtains the expressions of minimum correlation presented in the proposition.

The maximum correlation is obtained when the Bernoulli variables are comonotonic, i.e., when the number of simultaneous fatalities is maximized. In the two-agent world comonotonicity means that, in each state in which agent 2 dies, agent 1 dies as well. Because the maximum probability of simultaneous fatalities is equal to $\Pr(\tilde{x}_1 = 1, \tilde{x}_2 = 1) = \min(p_1, p_2) = p_2$, their maximum correlation equals

$$\frac{p_2 - p_1 p_2}{\sqrt{p_1(1-p_1)p_2(1-p_2)}} = \frac{\sqrt{p_2}\sqrt{1-p_1}}{\sqrt{p_1}\sqrt{1-p_2}}. \quad \square$$

One practical remark on Lemma 1 seems in order. The natural bounds of the correlation interval are only attained in the special case where $p_1 = p_2 = \frac{1}{2}$. This implies that for most binary risks the interval of attainable correlation is strictly narrower than $\rho \in [-1, 1]$.

3. Risk Equity and Its Implications

Following Keeney (1980) and the subsequent literature cited in the introduction, we define risk equity based on a Pigou–Dalton transfer in risk: a nonleaky transfer of probability mass from a more exposed to a less exposed individual, so that the transfer does not reverse the ranking of the two individuals in terms of their probability to die.[4]

Definition 3. We say a distribution of fatalities $\tilde{d}$ is more equitable than another distribution $\tilde{d}'$ iff a Pigou–Dalton transfer in risk δ from a more exposed agent 1 to a less exposed agent 2 would reduce the risk faced by agent 1 and raise the risk faced by agent 2 without switching their ranking in terms of absolute risk and without changing other individuals' risks. Formally, the probabilities of dying *before* the transfer are p_1 and p_2, with $p_1 > p_2$; *after* the transfer, the probabilities of dying are $p_1' = p_1 - \delta$ and $p_2' = p_2 + \delta$, where $0 \leqslant \delta \leqslant (p_1 - p_2)/2$.

Accordingly, a Pigou–Dalton transfer in risk always decreases the "gap" in risk between the agents 1 and 2 (Keeney 1980, Adler et al. 2014). This definition is intuitive, but also general since any mean-preserving contraction in the distribution of individual risks within society can be obtained through a series of Pigou–Dalton transfers. Our first objective is to extend Keeney's result to our more general definition of catastrophic risk. Proposition 2 asserts that, under the assumption of independent risks before and after the transfer, more risk equity always implies a more catastrophic distribution of fatalities.

Proposition 2. *Assume that the risks faced by N agents are independent. In that case, any Pigou–Dalton transfer in risk between two agents leads to a more catastrophic distribution of fatalities, if the risks to the N agents remain independent after the transfer.*

Proof. Given independence, we can focus on the risks affected by the transfer. Consider a change in the risks faced by agents 1 and 2, and denote $\tilde{y} := \tilde{x}_3 + \cdots + \tilde{x}_N$. We define $\mathsf{E}[f(\tilde{d})] = \mathsf{E}[f_1(\tilde{x}_1 + \tilde{x}_2)]$, with $f_1(x) = \mathsf{E}[f(x + \tilde{y})]$. Since under the assumption of independent risks f_1 is concave iff f is concave, the presence of $N-2$ independent agents does not affect the comparative statics analysis. From Proposition 1 and the equivalence between a more catastrophic and a more variable distribution of fatalities, we know that it suffices to show that a Pigou–Dalton transfer in risk between agents 1 and 2 increases the variance. This is always true because $\mathsf{var}[\tilde{d}'] - \mathsf{var}[\tilde{d}] = [(p_1 - \delta)(1 - p_1 + \delta) + (p_2 + \delta)(1 - p_2 - \delta)] - [p_1(1 - p_1) + p_2(1 - p_2)]$, which is positive for any $\delta \leqslant (p_1 - p_2)/2$ (and which is exactly the condition needed for the Pigou–Dalton transfer). $\square$

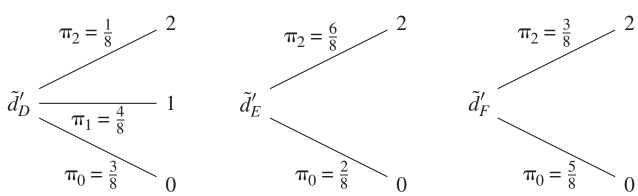
3.1. Two Agents Facing Dependent Risks

Our next objective is to extend the result of Proposition 2 to the case of dependent risks. To illustrate the complexity that arises from the interaction between only two risks ($N = 2$), we start from example (A) presented in Section 2.1 and analyze three possible risk transfers and their respective effect on the distribution of fatalities. Remember that in situation (A) the risks of two agents are independent. Here, and in contrast to Keeney (Proposition 2), we do not assume that the risks remain independent after the Pigou–Dalton transfer. Consider the following situations labeled (D), (E), and (F), respectively. (Risks after a transfer of $\delta = \frac{1}{8}$ are denoted by $\tilde{x}'_1$ and $\tilde{x}'_2$.) In all three situations, the agents face the same probability to die ($p'_1 := p_1 - \delta = p'_2 := p_2 + \delta = \frac{3}{8}$). However, in none of the situations the two risks are independent after the transfer (as the correlation ρ' is not equal to zero) and, consequently, Proposition 2 no longer applies:

$$\begin{array}{cc} \tilde{x}'_1 & \tilde{x}'_2 \end{array} \qquad \begin{array}{cc} \tilde{x}'_1 & \tilde{x}'_2 \end{array} \qquad \begin{array}{cc} \tilde{x}'_1 & \tilde{x}'_2 \end{array}$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(D) (E) (F)



The new correlation ρ' between the agents' risks can be computed as follows:

$$\rho' := \operatorname{corr}(\tilde{x}'_1, \tilde{x}'_2) = \frac{\mathsf{E}[\tilde{x}'_1\tilde{x}'_2] - (p_1 - \delta)(p_2 + \delta)}{\sqrt{(p_1-\delta)(1-p_1+\delta)}\sqrt{(p_2+\delta)(1-p_2-\delta)}}.$$

For the above situations, we have $\rho'_D = -0.0\overline{66}$, $\rho'_E = -0.6$, and $\rho'_F = 1$, respectively. The distribution of fatalities $\tilde{d}' := \tilde{x}'_1 + \tilde{x}'_2$ after each of the feasible Pigou–Dalton transfers is fully characterized by the corresponding probability trees. Compared to situation (A), the distribution of fatalities becomes strictly more catastrophic in situation (F), strictly less catastrophic in situation (E), and is identical in situation (D).

Situations (A) to (F) make it clear that the change in the distribution of fatalities is governed by two sources: (i) the effect of the Pigou–Dalton transfer in risk on the marginal distributions, and (ii) the change in correlation induced by the transfer. Both changes have an impact on how catastrophic the distribution of fatalities is. Situations (B) and (C) have illustrated that an increase (or decrease) in catastrophic risk might be caused by a change in correlation only. Situations (D) to (F) illustrate that an increase (or decrease) in catastrophic risk might also be due to a simultaneous change in correlation and risk equity.

The comparison of (A) with (D) is particularly revealing as (D) is obtained from (A) by switching a "1" and a "0" in a single row, and by relabeling the states. The distributions of fatalities are fully determined by the number of "1's" in each row. Therefore, the two distributions must be identical before and after the transfer so that $\tilde{d}_A$ and $\tilde{d}'_D$ have the same distribution. In other words, the induced change in the marginal distributions of individual risks $\tilde{x}_i$ is "counteracted" by its negative impact on the correlation, which keeps the sum of $\tilde{x}_i$'s identically distributed. This insight underlines that it is all but simple to extend Proposition 2, and there is no hope to generically sign the comparative statics analysis for any possible risk transfer without making specific restrictions on the correlation.[5]

3.2. The Necessary and Sufficient Condition

In this section we pin down the necessary and sufficient condition under which a Pigou–Dalton transfer in risk between two agents results in a more catastrophic distribution of fatalities. The condition comprises the special case in which the risks have identical correlation before and after the transfer. Yet we also provide a more general analysis, in which we allow changes in the correlation before and after the transfer. This degree of generality is important for many real-world applications.[6]

Next, we demonstrate that a Pigou–Dalton transfer in risk makes the distribution of fatalities more catastrophic whenever the correlation after the transfer is larger than a specified threshold.

Proposition 3. *Assume $N = 2$ and $p_1 > p_2$. Let ρ denote the correlation between the initial risks $\tilde{x}_1$ and $\tilde{x}_2$. After a Pigou–Dalton transfer in risk $\delta \in [0, (p_1 - p_2)/2]$, the distribution of fatalities becomes more catastrophic iff the correlation ρ' between $\tilde{x}'_1$ and $\tilde{x}'_2$ is larger than the critical level of correlation ρ^*, i.e.,*

$$\rho' \geq \rho^* := \frac{\delta(p_2 - p_1 + \delta) + \rho\sqrt{p_1}\sqrt{1-p_1}\sqrt{p_2}\sqrt{1-p_2}}{\sqrt{1-p_1+\delta}\sqrt{p_1-\delta}\sqrt{p_2+\delta}\sqrt{1-p_2-\delta}}. \quad (1)$$

Proof. We first extend a result in Proposition 1 showing that even when the marginals p_1 and p_2 are not kept fixed before and after the risk transfer, an increase in the probability of simultaneous deaths is equivalent to a more catastrophic distribution of fatalities. Let $\pi'_2 = \pi_2 + \gamma \geqslant 0$. Keeping the number of expected fatalities constant, i.e., $\pi'_1 + 2\pi'_2 = \pi_1 + 2\pi_2$, we have $\pi'_1 = \pi_1 - 2\gamma \geqslant 0$ and in turn $\pi'_0 = \pi_0 + \gamma$, because the sum of probabilities must equal one. Therefore, the distribution is more catastrophic iff $\gamma \geqslant 0$, and thus $\pi'_2 \geqslant \pi_2$.

Next, we compute the respective probabilities of simultaneous deaths before and after the risk transfer. We find that

$$\pi_2 = \mathsf{E}[\tilde{x}_1\tilde{x}_2] = p_1 p_2 + \rho\sqrt{p_1(1-p_1)}\sqrt{p_2(1-p_2)}, \quad (2)$$

and

$$\begin{aligned}\pi_2' &= \mathsf{E}[\tilde{x}_1'\tilde{x}_2'] \\ &= (p_1-\delta)(p_2+\delta) + \rho'\sqrt{(p_1-\delta)(1-p_1+\delta)} \\ &\quad\cdot\sqrt{(p_2+\delta)(1-p_2-\delta)}; \quad (3)\end{aligned}$$

A11 $\pi_2' \geqslant \pi_2$ whenever ρ' is sufficiently high (i.e., iff condition (1) is satisfied). □

Note that when $\rho = 0$, then $\rho^* < 0$. This result is intuitive as a Pigou–Dalton transfer when risks are independent increases the joint probability of deaths (see also Proposition 4, (i)). As stated above, a Pigou-Dalton transfer in risk may have two distinct effects on the distribution of fatalities: (i) through the change in the marginal distributions and (ii) through a change in the correlation between the two risks. Condition (1) in Proposition 3 depends on both effects, and it seems useful to think about them separately. The effect on the marginal distributions corresponds to a change in δ, keeping the correlation structure fixed: $\rho' = \rho$. The effect on the dependence structure corresponds to a change in correlation from ρ to ρ' assuming no changes in the marginal distributions (i.e., $\delta = 0$). However, it would be fallacious to separate the two effects as the change in correlation is bounded (see Lemma 1), and the range over which it is defined depends on the marginal distributions.[7] In other words, the range of attainable correlation between $\tilde{x}_1'$ and $\tilde{x}_2'$ is a function of δ, because the correlation ρ' between two risks with respective probabilities $p_1 - \delta$ and $p_2 + \delta$ cannot be larger than

$$\rho_{\max}(\delta) := \frac{\sqrt{p_2+\delta}}{\sqrt{p_1-\delta}}\frac{\sqrt{1-p_1+\delta}}{\sqrt{1-p_2-\delta}}. \quad (4)$$

The minimum correlation depends on whether $(p_1-\delta) + (p_2+\delta) = p_1 + p_2$ is larger than 1 or not and is also defined by Lemma 1. Furthermore, $\rho_{\max}(0)$ equals the maximum correlation ρ between the initial risks if $\delta = 0$.

Figure 1 displays the comparative statics analysis for a numerical example where the two effects are simultaneously at work. The grey-shaded areas in the four panels of Figure 1 represent the admissible range for the transfer δ (on the x-axis) and the correlation ρ' (on the y-axis) parameters such that the distribution of fatalities is more catastrophic after the Pigou–Dalton transfer (the example assumes that $p_1 = 0.8$, $p_2 = 0.3$, $\delta \in [0, (p_1-p_2)/2]$, and considers different initial values of ρ). More specifically, Figure 1 illustrates four situations in which the initial correlation between the two agents' risks is equal to $\rho = \rho_{\min}(0) = -0.76$ (panel A), $\rho = -0.5$ (panel B), $\rho = 0$ (panel C), and $\rho = \rho_{\max}(0) = 0.33$ (panel D), respectively. Each panel plots the critical level of correlation ρ^* as well as the attainable minimum and maximum correlation (4) as a function of δ using the relationships of Lemma 1. Two areas are displayed, the grey-hatched (light green) area corresponds to the case in which catastrophic risk increases (decreases) after the Pigou–Dalton transfer of δ.

Several special cases of Proposition 3 are worth being discussed in more detail. We summarize them in Proposition 4.

Proposition 4. *Assume $N = 2$ and $p_1 > p_2$. In the following special cases, the distribution of fatalities becomes more catastrophic after a Pigou–Dalton transfer in risk*:

(i) *$\rho' \geq \rho$ and $\delta > 0$ (which includes Keeney's 1980 result for $\rho' = \rho = 0$, and fixed correlation $\rho' = \rho$, as special cases)*;

(ii) *$\delta = 0$ and $\rho' > \rho$ (no change in the marginal distribution of fatalities, but an increase in correlation)*;

(iii) *ρ is equal to the minimum correlation (as computed in Lemma 1) between the two risks before the Pigou–Dalton transfer with $\delta > 0$*;

(iv) *ρ' is equal to the maximum correlation (as computed in Lemma 1) between the two risks after the Pigou–Dalton transfer with $\delta > 0$*;

(v) *$\delta > 0$ and $p_1 = 1$ (agent 1 is certain to die) or $p_2 = 0$ (agent 2 is certain to survive).*

Proof. We start from the expressions of π_2 and π_2' given by Equations (2)–(3) and show that $\pi_2' \geq \pi_2$ in each of the above statements (i)–(v).

To prove (i), note that by definition a Pigou–Dalton transfer imposes $\delta \leqslant (p_1-p_2)/2$, so that $p_1p_2 \leq (p_1-\delta)(p_2+\delta)$ and $\sqrt{1-p_1+\delta}\sqrt{p_1-\delta}\sqrt{p_2+\delta}\sqrt{1-p_2-\delta} > \sqrt{p_1}\sqrt{1-p_1}\sqrt{p_2}\sqrt{1-p_2}$. For $\rho' \geq \rho \geq 0$, the proof of (i) follows directly from the expressions of π_2 and π_2' in Equations (2)–(3) and $\pi_2' \geq \pi_2$. For $\rho \leq \rho' < 0$, the result still holds but the proof is longer and therefore relegated to the appendix.

The proof of (ii) follows directly from Proposition 1. This case isolates the effect of correlation.

Note that (iii) and (iv) are special cases of (i).

Finally, we prove (v) as follows. First, observe that for $p_1 = 1$ we have

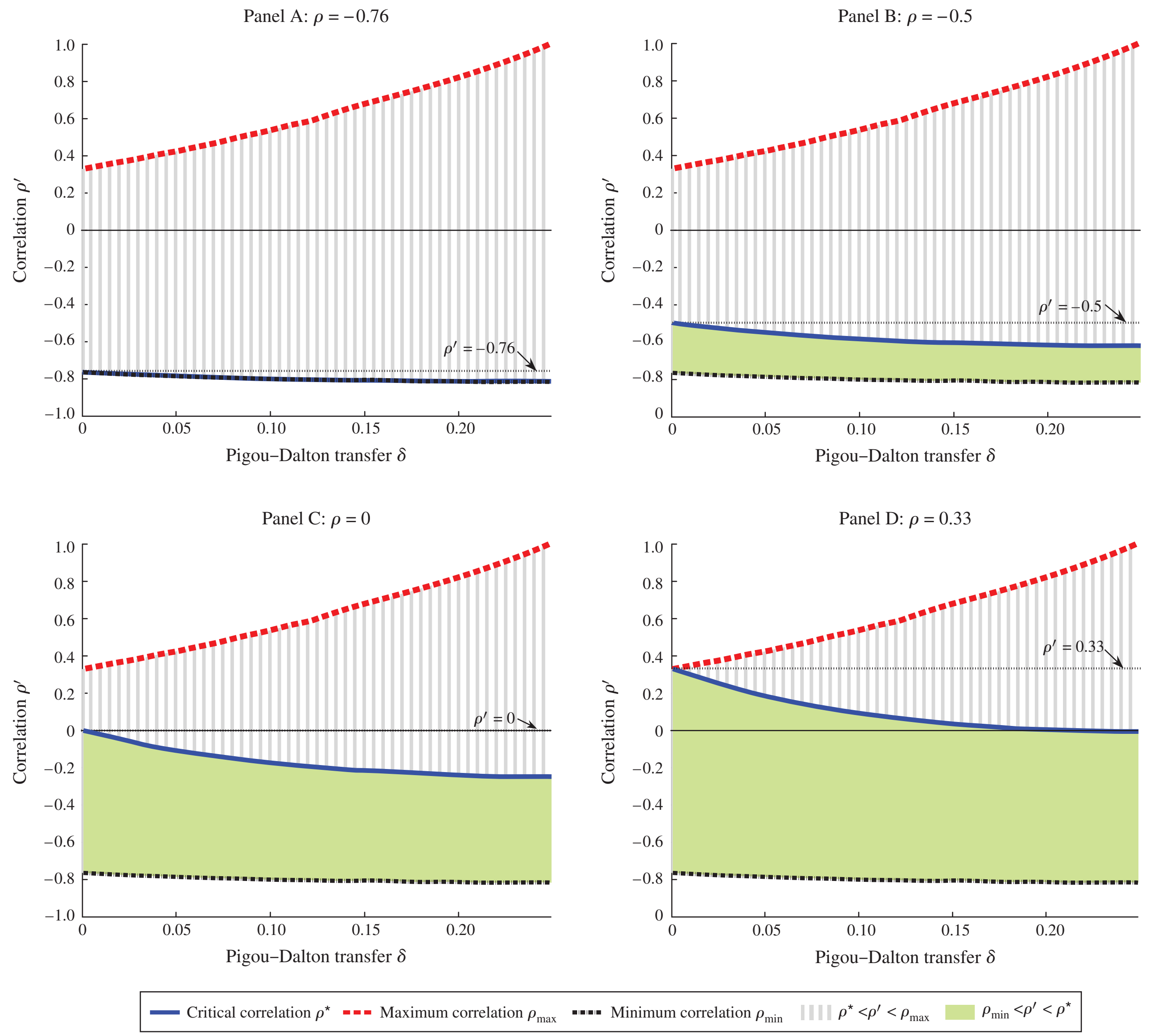
$$\rho^* := \delta(p_2 - 1 + \delta)/(\sqrt{\delta(1-\delta)}\sqrt{(p_2+\delta)(1-p_2-\delta)}),$$

which equals the minimum bound identified by Lemma 1 (for $p_1 + p_2 > 1$). Second, observe that for $p_2 = 0$,

$$\rho^* = \delta(-p_1+\delta)/(\sqrt{(1-p_1+\delta)(p_1-\delta)}\sqrt{\delta(1-\delta)}),$$

which equals the minimum bound identified by Lemma 1 (for $p_1 + p_2 \leq 1$). Thus, we always obtain $\rho' \geq \rho^*$ for these two special cases. □

Figure 1. (Color online) Correlation Domains of ρ' for Four Initial Values of ρ, Assuming $p_1 = 0.8$, $p_2 = 0.3$, and $\delta \in [0, (p_1 - p_2)/2]$



Statement (i) of Proposition 4 is apparent in all of the four panels in Figure 1: the horizontal line representing the level $\rho' = \rho$ always belongs to the grey-shaded area where the distribution of fatalities becomes more catastrophic. Statement (ii) follows immediately from the inspection of the ρ-values at $\delta = 0$. Statement (iii) corresponds to panel A in Figure 1, for which the correlation between the initial risks is minimum. Statement (iv) is, again, apparent in all four panels as the dashed line that represents the maximum correlation for ρ' always belongs to the grey-shaded area, in which the distribution of fatalities is more catastrophic. Statement (v) illustrates a Pigou–Dalton transfer between two agents in which the correlation between the initial risks is equal to 0, which is also the minimum attainable correlation.

4. Generalization to More Than Two Agents

We already observed in the two-agent world that the distribution of fatalities can become more or less catastrophic when either the correlation or the marginal distributions are altered through a Pigou–Dalton transfer in risk. In this section, we parallel the previous discussion for $N > 2$ agents. As in Section 2.2, we first discuss whether the distribution of fatalities becomes more (or less) catastrophic when only the dependence structure changes. Based on Section 3, we then look at the effect of a change in the marginal distributions of two risks in the presence of $N-2$ other agents. Under the assumption of a fixed dependence structure among the $N-2$ other agents, we show that the results obtained for $N=2$ still hold. Next, we relax the constraint on the dependence structure and, instead, make assumptions about pairwise correlations. As is well known, pairwise correlations alone do not provide sufficient information to pin down the dependence structure of the $\tilde{x}_1, \ldots, \tilde{x}_N$ risks. We can, however, show that the distribution of fatalities becomes more variable after a Pigou–Dalton transfer with uncorrelated risks. At the end of Section 4, we derive more general sufficiency conditions under which the distribution of fatalities becomes more variable after a Pigou–Dalton transfer in risk.

4.1. Extremal Dependence and the Distribution of Fatalities

In situations with $N > 2$ agents it is not obvious how one should compare distributions of fatalities against each other as it is unclear how the dependence structure among N risks should be defined. Even for Bernoulli random variables, the dependence structure involves more than the pairwise correlation coefficients ρ_{ij} (as in Section 2.2).[8] Without restricting the dependence structure of the N-agents' risks, we can derive results for the comonotonic and antimonotonic case, respectively. These two extreme dependence structures are defined as follows.

Definition 4. Consider $i = 1, 2, \ldots, N$ agents with $p_i \in (0,1)$ and let $\tilde{d} = \tilde{x}_1 + \cdots + \tilde{x}_N$ be the corresponding distribution of fatalities. Let then $\tilde{d}^c$ denote the comonotonic dependence structure (also known as the maximum correlation between the risks). Moreover, let $\tilde{d}^a$ denote the antimonotonic dependence structure implying that in all states either M or $M+1$ deaths occur. Formally, M is the integer number such that the expected number of fatalities is $\mu := p_1 + p_2 + \cdots + p_N \in [M, M+1[$. The distribution of fatalities $\tilde{d}^a$ thus takes one of two values: M with probability $p_M = M + 1 - \mu$, or $M+1$ with probability $1 - p_M = \mu - M$.[9]

Based on this definition, we obtain the following intuitive result.

Proposition 5. *The distribution of fatalities $\tilde{d}$ is always less catastrophic than $\tilde{d}^c$, and always more catastrophic than $\tilde{d}^a$. Namely, for all f concave and all possible distributions of fatalities $\tilde{d}$:*

$$\mathsf{E}[f(\tilde{d}^c)] \le \mathsf{E}[f(\tilde{d})] \le \mathsf{E}[f(\tilde{d}^a)] \\ = f(M)p_M + f(M+1)(1-p_M).$$

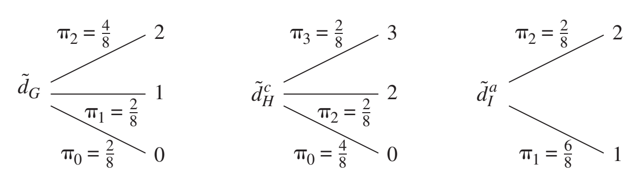
Proposition 5 is closely related to Lemma 1 as it defines an admissible range of dependence for $N > 2$ agents (see the appendix for a detailed proof). The most and least catastrophic distributions of fatalities, $\tilde{d}^c$ and $\tilde{d}^a$, are obtained when correlation is "maximized" and "minimized", respectively. Meilijson and Nadas (1979) proved that maximum correlation is obtained whenever risks are comonotonic; i.e., concentrated to specific states of the world. The least catastrophic distribution of fatalities is obtained by a generalization of the negative dependence structure in N dimensions. This generalization is far from trivial, however. Take the example of three risks—$\tilde{x}$, $\tilde{y}$, and $\tilde{z}$—and assume that $\tilde{x}$ is negatively correlated with $\tilde{y}$ and also negatively correlated with $\tilde{z}$; then, by definition, $\tilde{y}$ and $\tilde{z}$ must be positively correlated. This simple example highlights that it is not straightforward to define what it means that three variables are negatively correlated (Meyer and Strulovici 2012). In the special case of Proposition 5, all risks are Bernoulli distributed and the least catastrophic distribution of fatalities is therefore explicitly known (see Bernard et al. 2017 and the proof in the appendix). For more general distributions of risks, Puccetti and Rüschendorf (2012) recently proposed a rearrangement algorithm, which approximates the antimonotonic dependence structure of high-dimensional problems.[10]

Let us further illustrate Proposition 5. Situations (G) to (I) have the same marginal distributions for $\tilde{x}_1$, $\tilde{x}_2$, and $\tilde{x}_3$ ($p_1 = 1/2$, $p_2 = 1/4$, and $p_3 = 1/2$), but they differ in terms of their dependence structure:

$$\begin{array}{ccc} \tilde{x}_1 & \tilde{x}_2 & \tilde{x}_3 \end{array} \qquad \begin{array}{ccc} \tilde{x}_1 & \tilde{x}_2 & \tilde{x}_3 \end{array} \qquad \begin{array}{ccc} \tilde{x}_1 & \tilde{x}_2 & \tilde{x}_3 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(G) $\qquad$ (H) $\qquad$ (I)



Situation (G) is a situation with uncorrelated risks. Specifically, all pairs $\{\tilde{x}_1, \tilde{x}_2\}$, $\{\tilde{x}_1, \tilde{x}_3\}$, and $\{\tilde{x}_2, \tilde{x}_3\}$ have pairwise zero correlation ($\rho_{12} = \rho_{13} = \rho_{23} = 0$). Situation (H) gives rise to the most catastrophic distribution of fatalities $\tilde{d}_H^c$, and situation (I) to the least catastrophic distribution of fatalities $\tilde{d}_I^a$. The expected number of fatalities is in all three situations $\mu = \mathsf{E}[\tilde{d}] = 1.25$, but the range of possible outcomes differs across the situations. In particular, $\tilde{d}_I^a$ takes on only two values: one death with probability $p_M = 1 + 1 - 1.25 = 0.75$ and two deaths with probability $1 - p_M = 0.25$. Also observe that the least catastrophic distribution of fatalities is such that each individual risk $\tilde{x}_i$ is in reverse order with $\tilde{d}_{-i} := \sum_{j \neq i} \tilde{x}_j$ (i.e., with the distribution of fatalities over all but agent i). In other words, the states wherein $\tilde{x}_i = 1$ correspond to the states of the smallest value of $\tilde{d}_{-i}$ are attained. (As outlined in Endnote 10, this is necessary to attain minimum correlation.) Lastly, note that the variability of the distributions of fatalities largely differs across the situations: $\mathsf{var}(\tilde{d}_G) = 11/16$ in (G), $\mathsf{var}(\tilde{d}_H^c) = 27/16$ in (H), and $\mathsf{var}(\tilde{d}_I^a) = 3/16$ in (I). By definition, the latter two variance terms are the maximum and minimum variance, respectively.

The result in Proposition 5 indicates that the equivalence result between an increase in dependence and a more catastrophic distribution holds under $N > 2$ for the two most extreme distributions of fatalities. However, the result is not general because it does not characterize the effect of "more dependence." Moreover, we can easily show that another equivalence result of Proposition 1 fails. An increasing probability of simultaneous fatalities no longer implies a more catastrophic distribution. A simple counterexample with $N = 3$ suffices to demonstrate this. Define $\tilde{d}$ by $(\pi_0, \pi_1, \pi_2, \pi_3) = (0, 3/5, 0, 2/5)$ and $\tilde{d}'$ by $(\pi_0, \pi_1, \pi_2, \pi_3) = (1/5, 0, 3/5, 1/5)$. Both distributions have the same mean (i.e., 9/5), but $\tilde{d}$ implies a higher probability of simultaneous fatalities π_3 (i.e., $2/5 > 1/5$). Nevertheless, $\tilde{d}$ has a lower probability that at least two fatalities occur (i.e., $2/5 < 4/5$). There exist (at least) two concave functions that do not imply the same order: $\mathsf{E}[-\max(\tilde{d} - 1, 0)] = -0.8 > \mathsf{E}[-\max(\tilde{d}' - 1, 0)] = -1$ and $\mathsf{E}[-\max(\tilde{d} - 2, 0)] = -0.4 < \mathsf{E}[-\max(\tilde{d}' - 2, 0)] = -0.2$. Therefore, $\tilde{d}$ cannot be more catastrophic than $\tilde{d}'$.

4.2. Pigou–Dalton Transfers with Fixed Dependence

We now examine the impact of a Pigou–Dalton transfer in risk between two agents in a risky social situation involving $N > 2$ agents. Remember that the Keeney (1980) result considers N agents but assumes risk independence among the N agents. In practice, independent risks are often implausible and we therefore extend Keeney's result assuming *fixed* risk dependence among the $N - 2$ other agents, who are not involved in the Pigou–Dalton transfer.

Proposition 6. *Let $N > 2$, and consider a Pigou–Dalton transfer in risk between agents 1 and 2. Assume that this transfer does not affect the dependence between the other agents in the following sense: only the probabilities that either agent 1 or 2 dies and that both agents 1 and 2 die simultaneously may be altered by the risk transfer. Then, the result given in Proposition 3 still holds: the distribution of fatalities becomes more catastrophic iff $\rho' \geq \rho^*$ where ρ' is the correlation between agents 1 and 2's risks after the risk transfer and ρ^* is the critical level of correlation given in Equation (1).*

Proof. See the appendix.

Proposition 6 demonstrates that, if the transfer in risk does not affect the dependence structure among the $N - 2$ other agents, Keeney's result carries over to $N > 2$ agents provided it holds for agents 1 and 2 (in the $N = 2$ case). The latter condition is met, for instance, if the correlation between the two agents stays the same, i.e., $\rho = \rho'$, as shown in Proposition 4.

Another special case assumes *fixed* dependence among all N agents, which implies fixed pairwise correlations and, therefore, $\rho = \rho'$. As an immediate corollary of Proposition 6, Keeney's result holds whenever the dependence among all N agents is kept fixed. One may argue that keeping the dependence structure among the agents not involved in the Pigou–Dalton transfer fixed is a strong assumption. Yet, as we show in Section 4.4, if one relaxes this constraint, one might not even be able to conclude whether or not the resulting distribution of fatalities is more variable.

4.3. Pigou–Dalton Transfers with Pairwise Uncorrelated Risks

In the remainder of the paper, we will relax the constraint on the dependence structure, and make instead assumptions about pairwise correlations. Even though it is well known that pairwise correlations provide an insufficient statistic to map out the full dependence structure of multiple risks (Embrechts et al. 2002), economists and decision scientists often rely on correlation coefficients to measure the dependence between two variables.

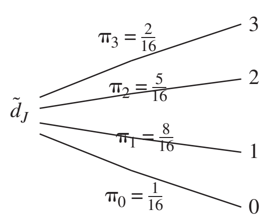
We start with a striking negative result. Indeed, we show that Keeney's result does not even hold when all the N risks are pairwise uncorrelated. Again, we make use of a simple example. Consider situations (J) and (K), below. The two situations consist of $S := 16$ states of the world. In each situation, three agents are faced with the risks $\tilde{x}_1$, $\tilde{x}_2$, and $\tilde{x}_3$, whose pairwise correlation coefficients are equal to zero before (J) and after the Pigou–Dalton transfer (K). One can easily verify that $\mathsf{corr}(\tilde{x}_1, \tilde{x}_2) = \mathsf{corr}(\tilde{x}_1, \tilde{x}_3) = \mathsf{corr}(\tilde{x}_2, \tilde{x}_3) = 0$ and

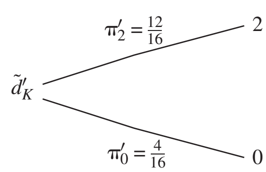
$\mathrm{corr}(\tilde{x}_1', \tilde{x}_2') = \mathrm{corr}(\tilde{x}_1', \tilde{x}_3) = \mathrm{corr}(\tilde{x}_2', \tilde{x}_3) = 0$. The distributions of fatalities, $\tilde{d}_J$ and $\tilde{d}_K'$, are again depicted as trees:

$$\begin{array}{ccc} \tilde{x}_1 & \tilde{x}_2 & \tilde{x}_3 \end{array} \qquad\qquad \begin{array}{ccc} \tilde{x}_1' & \tilde{x}_2' & \tilde{x}_3' \end{array}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad\qquad \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$(J) \qquad\qquad (K)$





Observe that the variance increases from $\mathrm{var}(\tilde{d}_J) = \frac{5}{8}$ to $\mathrm{var}(\tilde{d}_K') = \frac{6}{8}$ because of the Pigou–Dalton transfer between agent 1 and 2, demonstrating that the post-transfer distribution of risk is more variable. However, $\tilde{d}_K'$ is not more catastrophic. Consider the concave function $f(x) = -\max(x-\psi, 0)$. For $\psi = 2.5$, $\mathrm{E}[f(\tilde{d}_K')] = 0 > \mathrm{E}[f(\tilde{d}_J)]$. Therefore, the Pigou–Dalton transfer results in an increased variability, but not in more catastrophic risk.

Situations (J) and (K) indicate that pairwise zero correlation is insufficient to maintain Keeney's result when risks are dependent. Of course, pairwise zero correlation does not imply independence. In the above example, $\tilde{x}_1$, $\tilde{x}_2$, and $\tilde{x}_3$ are not independent:

$$\Pr(\tilde{x}_1 = 1, \tilde{x}_2 = 1, \tilde{x}_3 = 1) = \frac{2}{16} \neq p_1 p_2 p_3 = \frac{3}{32}.$$

Neither are $\tilde{x}_1'$, $\tilde{x}_2'$, and $\tilde{x}_3$:

$$\Pr(\tilde{x}_1'=1, \tilde{x}_2'=1, \tilde{x}_3=1) = 0 \neq p_1' p_2' p_3 = (p_1-\delta)(p_2+\delta)p_3 = \frac{1}{8}.$$

We can then prove the following result.

Proposition 7. *Assume there are $N > 2$ agents facing the risks $\tilde{x}_1, \ldots, \tilde{x}_N$, all of which exhibit pairwise zero correlation, i.e., $\mathrm{corr}(\tilde{x}_i, \tilde{x}_j) = 0$ for all $i \neq j$. Moreover, assume that after a Pigou–Dalton transfer in risk between agent 1 and 2 the pairwise correlations are still equal to zero: $\mathrm{corr}(\tilde{x}_a', \tilde{x}_b') = 0$ for all $a \neq b$. Then,*

(i) *the distribution of fatalities after the Pigou–Dalton transfer may or may not be more catastrophic;*

(ii) *the distribution of fatalities after the Pigou–Dalton transfer is more variable.*

Proof. The counterexample shown above proves (i). To prove (ii), we compute the variance of the distribution of fatalities before and after the Pigou–Dalton transfer for a situation in which all risks have pairwise zero correlation. We assume $N = 3$. Before the transfer, the variance of the distribution of fatalities is given by

$$\mathrm{var}(\tilde{d}) = \mathrm{var}(\tilde{x}_1) + \mathrm{var}(\tilde{x}_2) + \mathrm{var}(\tilde{x}_3) + 2\rho\sqrt{\mathrm{var}(\tilde{x}_1)}\sqrt{\mathrm{var}(\tilde{x}_2)} + 2\mathrm{cov}(\tilde{x}_1, \tilde{x}_3) + 2\mathrm{cov}(\tilde{x}_2, \tilde{x}_3), \quad (5)$$

where $\mathrm{var}(\tilde{x}_i) = p_i(1-p_i)$ for $i = \{1,2,3\}$. After the transfer, the variance of the distribution of fatalities becomes

$$\mathrm{var}(\tilde{d}') = \mathrm{var}(\tilde{x}_1') + \mathrm{var}(\tilde{x}_2') + \mathrm{var}(\tilde{x}_3') + 2\rho'\sqrt{\mathrm{var}(\tilde{x}_1')}\sqrt{\mathrm{var}(\tilde{x}_2')} + 2\mathrm{cov}(\tilde{x}_1', \tilde{x}_3') + 2\mathrm{cov}(\tilde{x}_2', \tilde{x}_3'), \quad (6)$$

with $\mathrm{var}(\tilde{x}_1') = (p_1-\delta)(1-p_1+\delta)$, $\mathrm{var}(\tilde{x}_2') = (p_2+\delta)(1-p_2-\delta)$, $\mathrm{var}(\tilde{x}_3') = p_3(1-p_3)$.

Using the information on the pairwise zero correlation, we further simplify expressions (5) and (6) to

$$\mathrm{var}(\tilde{d}) = p_1(1-p_1) + p_2(1-p_2) + \mathrm{var}(\tilde{x}_3),$$
$$\mathrm{var}(\tilde{d}') = (p_1-\delta)(1-p_1+\delta) + (p_2+\delta)(1-p_2-\delta) + \mathrm{var}(\tilde{x}_3').$$

The result underlines that the Pigou–Dalton transfer only affects the variances of $\tilde{x}_1$ and $\tilde{x}_2$, while $\mathrm{var}(\tilde{x}_3') = \mathrm{var}(\tilde{x}_3)$ because the distribution $\tilde{x}_3$ did not change. It follows that for all $\delta \in [0, (p_1-p_2)/2]$, $\mathrm{var}(\tilde{d}') > \mathrm{var}(\tilde{d})$. The generalization of the proof to $N > 3$ is straightforward. □

Proposition 7 demonstrates that, for $N > 2$, it is impossible to make generic statements about whether the distribution becomes more or less catastrophic based on pairwise correlation coefficients alone. Yet one can conclude on the variability of the distribution of fatalities. In the rest of the paper, we will thus focus our attention on the variability of the distribution of fatalities.

4.4. Pigou–Dalton Transfers with Correlated Risks

In this section, we address situations in which the risks to $N > 2$ agents are correlated. A Pigou–Dalton transfer in risk is then implemented between agents 1 and 2 (presuming $p_1 > p_2$). If there is dependence among the risks in a N-agent world, one cannot know whether or not the distribution of fatalities becomes more variable after the Pigou–Dalton transfer. The following impossibility result underpins our claim.

Proposition 8. *Assume there are $N > 2$ agents facing the risks $\tilde{x}_1, \ldots, \tilde{x}_N$. The effect of a Pigou–Dalton transfer in risk (from agent 1 to agent 2) on the distribution of fatalities is ambiguous in the following sense: if the dependence structure among agents $3, \ldots, N$ is altered by the transfer, and if N is large enough, then it is generally impossible to conclude about whether the distribution of fatalities becomes more or less variable.*

Proof. See the appendix.

Proposition 8 states that it is impossible to predict how a Pigou–Dalton transfer in risk alters the degree of variability when the transfer in risk can arbitrarily affect the dependence structure of agents not directly involved in the transfer. In the following, we further constrain the problem and assume that the variance of the sum of risks faced by the agents not involved in the transfer is fixed. This additional constraint allows us to formally describe how a Pigou–Dalton transfer between two correlated risks affects the distribution of fatalities. While the equivalence results established in Proposition 9 resemble those established in Proposition 6, it is important to recall that, in general, the notion of "more variable" is less demanding than that of "more catastrophic."

Proposition 9. *Assume that there are $N > 2$ agents facing the risks $x_1, \ldots, x_N$ and the variance of the distribution of fatalities of agents $3, \ldots, N$ is not altered through a Pigou–Dalton transfer in risk $\delta \in [0, (p_1 - p_2)/2]$ between agent 1 and agent 2. Then, the two following statements are equivalent:*

(i) *the distribution of fatalities after the Pigou–Dalton transfer is more variable;*

(ii) *the new correlation ρ' between $\tilde{x}_1'$ and $\tilde{x}_2'$ is strictly larger than a critical level of correlation, i.e.,*

$$\rho' \geq \rho^* + \frac{\mathrm{cov}\left(\tilde{x}_1 + \tilde{x}_2, \sum_{i=3}^{N} \tilde{x}_i\right) - \mathrm{cov}(\tilde{x}_1' + \tilde{x}_2', \sum_{i=3}^{N} \tilde{x}_i)}{\sqrt{(p_1 - \delta)(1 - p_1 + \delta)}\sqrt{(p_2 + \delta)(1 - p_2 - \delta)}}, \tag{7}$$

where ρ^* is the critical level of correlation (1) given in Proposition 3 for the two-agent world.

Proof. By assumption $\mathrm{var}(\tilde{y}) = \mathrm{var}(\tilde{y}')$. Therefore, $\mathrm{var}(\tilde{d}') > \mathrm{var}(\tilde{d})$ iff

$$\mathrm{var}(\tilde{x}_1') + \mathrm{var}(\tilde{x}_2') + 2\rho'\sqrt{\mathrm{var}(\tilde{x}_1')}\sqrt{\mathrm{var}(\tilde{x}_2')} + 2\mathrm{cov}(\tilde{x}_1' + \tilde{x}_2', \tilde{y})$$
$$> \mathrm{var}(\tilde{x}_1) + \mathrm{var}(\tilde{x}_2) + 2\rho\sqrt{\mathrm{var}(\tilde{x}_1)}\sqrt{\mathrm{var}(\tilde{x}_2)} + 2\mathrm{cov}(\tilde{x}_1 + \tilde{x}_2, \tilde{y}).$$

Solving for ρ' yields

$$\rho' > \frac{\delta(\delta - p_1 + p_2)}{\sqrt{\mathrm{var}(\tilde{x}_1')}\sqrt{\mathrm{var}(\tilde{x}_2')}} + \rho\frac{\sqrt{\mathrm{var}(\tilde{x}_1)}\sqrt{\mathrm{var}(\tilde{x}_2)}}{\sqrt{\mathrm{var}(\tilde{x}_1')}\sqrt{\mathrm{var}(\tilde{x}_2')}} + \frac{\mathrm{cov}(\tilde{x}_1 + \tilde{x}_2 - \tilde{x}_1' - \tilde{x}_2', \tilde{y})}{\sqrt{\mathrm{var}(\tilde{x}_1')}\sqrt{\mathrm{var}(\tilde{x}_2')}},$$

where the first two terms on the right-hand side are equal to the critical level of correlation ρ^* found in Proposition 3 for $N = 2$. □

Some observations on Proposition 9 are warranted. First, note that for the special case analyzed by Keeney (1980) we have $\rho = \rho' = \mathrm{cov}(\tilde{x}_1 + \tilde{x}_2 - \tilde{x}_1' - \tilde{x}_2', \tilde{y}) = 0$ so that the inequality (7) is satisfied, because $\delta - p_1 + p_2 < 0$. More generally, the extension to N agents does not yield a different result than the one obtained for two agents when

$$\mathrm{cov}\left(\tilde{x}_1 + \tilde{x}_2, \sum_{i=3}^{N} \tilde{x}_i\right) = \mathrm{cov}\left(\tilde{x}_1' + \tilde{x}_2', \sum_{i=3}^{N} \tilde{x}_i\right).$$

In words, Proposition 9 holds whenever the Pigou–Dalton transfer in risk between agent 1 and agent 2 does not affect the correlation between the distribution of fatalities of these two agents and the distribution of fatalities of the other $N-2$ agents. This condition is less restrictive than assuming (as we do in Proposition 6) that the dependence of among the agents is fixed. On the other hand, Proposition 9 considers a weaker concept than that of more catastrophic.

5. Conclusion

Risks that may cause many thousand casualties are ubiquitous in today's world. This paper examines the statistical dependence structure of risky social situations. In particular, we explore the relationship between more catastrophic and more equitable distributions of risk. To do so, we define a more catastrophic situation as a mean-preserving spread in the distribution of fatalities, and a more equitable situation as one that results in a smaller difference between the probabilities of death faced by any two agents. Based on these definitions, we demonstrate that more correlation between two risks is equivalent to a higher probability of simultaneous deaths and we characterize a set of conditions under which more equity induces such a more catastrophic situation.

The key contribution of our paper is the extension of the Keeney (1980) seminal result on the ex ante/ex post conflict in risk management to a world in which risks are interdependent. It delivers a simple message: Keeney's result holds whenever the dependence structure is not altered "too much" by the risk transfer, implying that it always holds when the dependence structure remains unchanged. The result breaks down when the transfer leads to a significant change in how the total risk burden is shared among the population at risk. This limitation is important as it implies that, in specific situations, a policy-induced redistribution of risk can simultaneously reduce inequity and alleviate catastrophe risk. In general, however, a trade-off between the two objectives is to be made. Yet, common social welfare functions (utilitarian, maximin, prioritarian) are not flexible enough to consider the trade-off between ex ante and ex post objectives (Bovens and Fleurbaey 2012, Rheinberger and Treich 2016)). Policy makers who wish to rank risky social situations with respect to these objectives may therefore have to fall back on different approaches such as multicriteria analysis.

Generalizations to more than two agents come at the cost of additional complexity. It is impossible to obtain any result for multiple risks without specifying the change in the dependence structure. The characterization of all pairwise correlations is—perhaps unsurprisingly—not enough to derive results on the dependence structure of multiple risks. Keeney's result does not even hold when the risks are pairwise uncorrelated; it is sufficient, however, to conclude about the impact of the risk transfer between two agents—in the presence or absence of other agents—if one is interested in the weaker notion of more variable risk. In this case, the result applies whenever the risk transfer does not affect the correlation between the risk distributions of the two agents involved in the transfer and the distribution of risk among the other $N-2$ agents.

While we have derived the above results in the context of mortality risk, the underlying mathematics apply to other managerial decisions that involve risky binary outcomes. One might think of defining the optimal vaccination strategy, or allocating resources to innovation initiatives, or making investment decisions in pharmaceutical labs, to name only a few areas of application. There is mounting experimental evidence that people care about ex ante and ex post trade-offs in risky social decisions, and that the correlation of individual risks matters. We conclude that the analysis of the dependence structure of social risks is subtle and deserves more attention in future theoretical and empirical studies.

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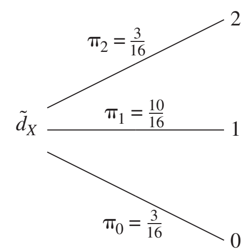
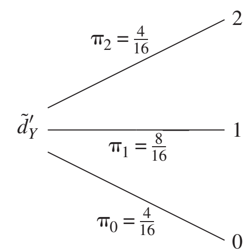
Appendix

A.1. Example of Independent Risks After the Risk Transfer

We asserted in Section 3 that it is possible to find examples in which the two risks before the Pigou–Dalton transfer in risk, $\tilde{x}_1$ and $\tilde{x}_2$, as well as after the transfer, $\tilde{x}'_1$ and $\tilde{x}'_2$, are uncorrelated and hence independent.[11] Consider the below situations (X) and (Y) with $S := 16$ states $(\omega_1, \ldots, \omega_{16})$:

$$
\begin{array}{cc}
\begin{array}{cc} \tilde{x}_1 & \tilde{x}_2 \end{array} & \begin{array}{cc} \tilde{x}'_1 & \tilde{x}'_2 \end{array} \\
\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} &
\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \\ 0 & 1 \\ 1 & 1 \\ 1 & 1 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \\
(X) & (Y)
\end{array}
$$





The individual risks $\tilde{x}_1$ and $\tilde{x}_2$ in the initial situation (X) are independent, and so are the individual risks $\tilde{x}'_1$ and $\tilde{x}'_2$ after a Pigou–Dalton transfer of $\delta = \frac{1}{4}$. When the risks are independent before and after the Pigou–Dalton transfer, as in situations (X) and (Y), the distribution of fatalities $\tilde{d}' = \tilde{x}'_1 + \tilde{x}'_2$ is more catastrophic than $\tilde{d} = \tilde{x}_1 + \tilde{x}_2$ as predicted by Keeney (Proposition 2).

A.2. Proof of Statement (i) in Proposition 4 When $\rho < 0$

Proof. From (1) we have that

$$\rho^*(\delta) := \frac{\delta(p_2 - p_1 + \delta) + \rho\sqrt{p_1}\sqrt{1-p_1}\sqrt{p_2}\sqrt{1-p_2}}{\sqrt{1-p_1+\delta}\sqrt{p_1-\delta}\sqrt{p_2+\delta}\sqrt{1-p_2-\delta}}.$$

Using Proposition 3, we need to show that $\rho' \geq \rho^*(\delta)$ for all $\delta \in (0, (p_1 - p_2)/2)$. As we assume that $\rho' \geq \rho$, it is enough to show that $\rho \geq \rho^*(\delta)$. Note that $\rho^*(0) = \rho$, thus we study the sensitivity of $\rho^*(\delta)$ to δ. We first compute the derivative of $\rho^*(\delta)$ with respect to δ:

$$\frac{\partial \rho^*(\delta)}{\partial \delta} = \frac{(p_1 - p_2 - 2\delta)(A(\delta)\rho + B(\delta))}{2[(p_1-\delta)(1-p_1+\delta)(1-p_2-\delta)(p_2+\delta)]^{3/2}}, \quad \text{(A.1)}$$

where

$$A(\delta) := \sqrt{p_1(1-p_1)p_2(1-p_2)}(2\delta^2 + 2\delta(p_2 - p_1) - (1 + 2p_1p_2 - p_2 - p_1))$$

and

$$B(\delta) := (p_1p_2 + (1-p_1)(1-p_2))(\delta^2 + \delta(p_2 - p_1) - 2p_1p_2) + 2p_1^2p_2^2.$$

Observe then that the sign of $\delta \mapsto \partial\rho^*(\delta)/\partial\delta$ over $[0,(p_1-p_2)/2]$ is the same as the sign of the function $\delta \mapsto A(\delta)\rho + B(\delta)$ for $\delta \in (0,(p_1-p_2)/2)$.

Lemma 2. *When $\rho < 0$, the function $\delta \mapsto A(\delta)\rho + B(\delta)$ satisfies the following property*

$$\forall\, \delta \in (0, \tfrac{1}{2}(p_1 - p_2)), \quad A(\delta)\rho + B(\delta) < 0. \tag{A.2}$$

Proof of Lemma 2. From the expressions of $A(\delta)$ and $B(\delta)$, we know that $A(\delta)\rho + B(\delta)$ is a second-degree polynomial of δ. Differentiating $\delta \mapsto A(\delta)\rho + B(\delta)$ gives an affine equation in δ and solving for its zero gives $\delta = (p_1 - p_2)/2$. The function $A(\delta)\rho + B(\delta)$ achieves its minimum at this value. In addition, we now prove that

$$A(0)\rho + B(0) < 0 \quad A'(0)\rho + B'(0) < 0. \tag{A.3}$$

Thus (A.2) follows because of the properties of a polynomial of the second degree in δ (it is decreasing over the interval $[0,(p_1-p_2)/2]$ and thus takes only negative values over this interval).

To prove (A.3), we need the expressions of $A(0)$, $B(0)$, $A'(0)$ and $B'(0)$:

$$\begin{aligned} A(0) &= -\sqrt{p_1(1-p_1)p_2(1-p_2)}(p_1p_2 + (1-p_1)(1-p_2)) \\ B(0) &= -2p_1p_2(1-p_1)(1-p_2) \\ A'(0) &= 2(p_2 - p_1)\sqrt{p_1(1-p_1)p_2(1-p_2)} \\ B'(0) &= (p_2 - p_1)(p_1p_2 + (1-p_1)(1-p_2)). \end{aligned}$$

We distinguish two cases:

Case 1: When $p_1 + p_2 \le 1$. Then from Lemma 1, $\rho \ge -\sqrt{p_1p_2}/\sqrt{1-p_1}\sqrt{1-p_2}$ then

$$\begin{aligned} A(0) + \rho B(0) &\le p_1p_2(p_1 + p_2 - 1) \le 0, \\ A'(0) + \rho B'(0) &\le (p_2 - p_1)(1 - p_1 - p_2) < 0, \end{aligned}$$

because $p_2 < p_1$.

Case 2: When $p_1 + p_2 > 1$. Then from Lemma 1, $\rho \ge -\sqrt{1-p_1}\sqrt{1-p_2}/\sqrt{p_1p_2}$ then

$$\begin{aligned} A(0) + \rho B(0) &\le (1-p_1)(1-p_2)(1 - p_1 + p_2) < 0, \\ A'(0) + \rho B'(0) &\le (p_2 - p_1)(p_1 + p_2 - 1) < 0, \end{aligned}$$

because $p_2 < p_1$. □

Proof of Statement (i) in Proposition 4 When $\rho < 0$. Since $\delta \le (p_1 - p_2)/2$, then from the expression (A.1) and Lemma 2, it is clear that

$$\forall\, \delta \in [0, \tfrac{1}{2}(p_2 - p_1)] \quad \frac{\partial\rho^*(\delta)}{\partial\delta} \le 0. \tag{A.4}$$

When $\delta = 0$, then $\rho^* = \rho$. Using the fact that ρ^* is decreasing in δ (i.e., (A.4)), then for all $\delta \in [0,(p_2-p_1)/2]$ $\rho^*(\delta) \le \rho^*(0) = \rho \le \rho'$. Since ρ' satisfies (1), then the distribution of fatalities is more catastrophic and (i) is proved. □

A.3. Proof of Proposition 5

The fact that the distribution of fatalities $\tilde{d}$ is always less catastrophic than $\tilde{d}^c$ was first proved by Tchen (1980). The proof of the first side of the inequality in Proposition 5 follows. The proof of the other side of the inequality is similar to that of Lemma 3.1 in Bernard et al. (2017). We use here the fact that the resulting distribution is less catastrophic and not only less variable as it is the case in their paper.

Lemma 3 (Least Catastrophic Distribution of Fatalities)**.** *Denote by μ the average number of fatalities. Define for $j = 1, \ldots, N$,*

$$a_j = \left(\sum_{i=1}^{j} p_i\right) \bmod 1,$$

and the sets

$$I_j = \begin{cases} [a_{j-1}, a_j] & \text{if } a_j > a_{j-1}, \\ [0, a_j] \cup [a_{j-1}, 1] & \text{if } a_j < a_{j-1}, \end{cases}$$

with the convention that $a_0 = 0$. Then, the least catastrophic distribution is $\tilde{d}^a := \sum_{j=1}^{N} \tilde{y}_j$ where $\tilde{y}_j$ are defined by

$$\tilde{y}_j = \mathbb{1}_{\tilde{u} \in I_j}, \tag{A.5}$$

where $\tilde{u}$ is a standard uniformly distributed random variable over $(0,1)$. *Furthermore, $\tilde{d}^a$ takes only two values M with probability $p_M = M + 1 - \mu$ and $M + 1$ with probability $1 - p_M$ where $M = \lfloor \mu \rfloor$ (largest integer inferior or equal to μ).*

Proof. Let us first observe that $\tilde{y}_j$ defined by (A.5) are Bernoulli with parameter p_j. Furthermore, $\tilde{d}^a = \tilde{y}_1 + \tilde{y}_2 + \cdots + \tilde{y}_N$ only takes values M with probability p_M or $(M+1)$ with probability $1 - p_M$ (where $p_M = 1$ may hold if it is constant). Consider any other distribution of fatalities $\tilde{d} = \tilde{x}_1 + \tilde{x}_2 + \cdots + \tilde{x}_N$ with $\tilde{x}_j$ being a Bernoulli distribution with parameter p_j and let us show that $\tilde{d}$ is more catastrophic than $\tilde{d}^a$. Observe that any such distribution of fatalities $\tilde{d}$ takes values in $\{0, 1, 2, \ldots, N\}$.

It is clear that $\forall\, x \in]0, M[$, $F_{\tilde{d}}(x) \ge F_{\tilde{d}^a}(x) = 0$ and $\forall\, x \in [(M+1), +\infty[$, $F_{\tilde{d}}(x) \le F_{\tilde{d}^a}(x) = 1$. Since $F_{\tilde{d}}(x)$ and $F_{\tilde{d}^a}(x)$ are constant on the interval $[M, M+1[$ one has

$$\exists\, c \geqslant 0, \quad \begin{cases} \forall\, x \in (0, c) & F_{\tilde{d}}(x) \ge F_{\tilde{d}^a}(x), \\ \forall\, x \in (c, +\infty) & F_{\tilde{d}}(x) \le F_{\tilde{d}^a}(x), \end{cases} \tag{A.6}$$

namely, $c = M + 1$ if $F_{\tilde{d}}(M) > F_{\tilde{d}^a}(x)$ and $c = M$ if $F_{\tilde{d}}(M) \le F_{\tilde{d}^a}(x)$. In other words, the distribution function $F_{\tilde{d}}$ crosses $F_{\tilde{d}^a}$ exactly once from above. Since $\mathsf{E}[\tilde{d}] = \mathsf{E}[\tilde{d}^a]$ this implies the well-known one-crossing property that characterizes second-order stochastic dominance. □

A.4. Proof of Proposition 6

Proof. The following notation is useful to demonstrate this result. Let Θ be a subset of $\{1, 2, \ldots, N\}$ agents and p_Θ be the probability that exactly this subset of agents die. The probability that agent i dies can then be rewritten as

$$p_i = p_{\{i\}} + \sum_{k_1 \ne i} p_{\{i,k_1\}} + \sum_{k_1,k_2 \ne i} p_{\{i,k_1,k_2\}} + \cdots + p_{\{1,2,\ldots,N\}} \tag{A.7}$$

because agent i can die alone or together with $k = 1, \ldots, N-1$ other agents.

We demonstrate the result for $N = 3$. Using Definition (A.7), we have

$$\begin{aligned}
p_1 &= p_{\{1\}} + p_{\{1,2\}} + p_{\{1,3\}} + p_{\{1,2,3\}},\\
p_2 &= p_{\{2\}} + p_{\{1,2\}} + p_{\{2,3\}} + p_{\{1,2,3\}},\\
p'_1 &= p'_{\{1\}} + p'_{\{1,2\}} + p'_{\{1,3\}} + p'_{\{1,2,3\}},\\
p'_2 &= p'_{\{2\}} + p'_{\{1,2\}} + p'_{\{2,3\}} + p'_{\{1,2,3\}}.
\end{aligned}$$

Let us now apply a Pigou–Dalton transfer in risk: $p'_1 = p_1 - \delta$ and $p'_2 = p_2 + \delta$, where $0 \leqslant \delta \leqslant (p_1 - p_2)/2$. Because of the assumption of fixed dependence, the Pigou–Dalton transfer in risk between agents 1 and 2 can only affect $p'_{\{1\}}$, $p'_{\{2\}}$ and $p'_{\{1,2\}}$. Let the probability that agents 1 and 2 die simultaneously become $p'_{\{1,2\}} = p_{\{1,2\}} + \gamma$. This leads to $p_1 - \delta = p'_{\{1\}} + p_{\{1,2\}} + \gamma + p'_{\{1,3\}} + p'_{\{1,2,3\}}$, which using $p_{\{1,3\}} = p'_{\{1,3\}}$ and $p_{\{1,2,3\}} = p'_{\{1,2,3\}}$ because of the fixed dependence assumption, implies

$$p'_{\{1\}} = p_{\{1\}} - \delta - \gamma.$$

Similarly, we have $p_2 + \delta = p'_{\{2\}} + p_{\{1,2\}} + \gamma + p'_{\{2,3\}} + p'_{\{1,2,3\}}$, which using again $p_{\{2,3\}} = p'_{\{2,3\}}$ and $p_{\{1,2,3\}} = p'_{\{1,2,3\}}$ implies

$$p'_{\{2\}} = p_{\{2\}} + \delta - \gamma.$$

Therefore, the probability of exactly one death after the Pigou–Dalton transfer is $\pi'_1 = p'_{\{1\}} + p'_{\{2\}} + p'_{\{3\}}$, which using $p'_{\{3\}} = p_{\{3\}}$, implies

$$\pi'_1 = \pi_1 - 2\gamma.$$

Similarly, the probability of exactly two deaths is $\pi'_2 = p'_{\{1,2\}} + p'_{\{1,3\}} + p'_{\{2,3\}}$, which using $p'_{\{1,3\}} = p_{\{1,3\}}$ and $p'_{\{1,2\}} = p_{\{1,2\}} + \gamma$, implies

$$\pi'_2 = \pi_2 + \gamma.$$

Finally, the probability that all $N = 3$ agents die simultaneously does not change because of the assumption of fixed dependence of agent 3's risk:

$$\pi'_3 = p'_{\{1,2,3\}} = p_{\{1,2,3\}} = \pi_3.$$

This further implies that the probability that nobody dies is equal to

$$\begin{aligned}
\pi'_0 &= 1 - \pi'_1 - \pi'_2 - \pi'_3\\
&= 1 - (\pi_1 - 2\gamma) - (\pi_2 + \gamma) - \pi_3\\
&= \pi_0 + \gamma.
\end{aligned}$$

The expressions above for π'_0, π'_1, π'_2 and π'_3 permit to conclude that the distribution of fatalities becomes more catastrophic after the transfer in risk iff $\gamma \geq 0$, namely, iff $\pi'_2 \geq \pi_2$. We may follow the proof of Proposition 3 to demonstrate the result. The proof for $N > 3$ is analogous. □

A.5. Proof of Proposition 8

Proof. We study the distribution of fatalities $\tilde{d} = \tilde{x}_1 + \tilde{x}_2 + \cdots + \tilde{x}_N$. For notational convenience we partition the distribution into $\tilde{d} = \tilde{x}_1 + \tilde{x}_2 + \tilde{y}$ with $\tilde{y} := \tilde{x}_3 + \cdots + \tilde{x}_N$. It suffices to show that it is not possible to conclude about whether the distribution becomes more variable when N is large enough. To do so, we compute the variance before and after the Pigou–Dalton risk transfer. The variance of the pretransfer distribution of fatalities, $\mathsf{var}(\tilde{d})$, and the posttransfer distribution of fatalities, $\mathsf{var}(\tilde{d}')$, follows from (5) and (6), respectively. We are interested in the change in variance:

$$\mathsf{var}(\tilde{d}') - \mathsf{var}(\tilde{d}) = \Delta_{\mathrm{PD}} + \Delta_{\mathrm{OA}} + 2\Delta_{\mathsf{cov}},$$

which we split into three terms.

The first term is the change in variance caused by the Pigou–Dalton transfer:

$$\Delta_{\mathrm{PD}} := \mathsf{var}(\tilde{x}_1{}' + \tilde{x}_2{}') - \mathsf{var}(\tilde{x}_1 + \tilde{x}_2);$$

the second term captures the change in dependence among the other agents not involved in the Pigou–Dalton transfer:

$$\Delta_{\mathrm{OA}} := \mathsf{var}(\tilde{y}') - \mathsf{var}(\tilde{y});$$

and the third term captures the change in dependence between the two agents involved in the Pigou–Dalton transfer and all the others:

$$\Delta_{\mathsf{cov}} := \mathsf{cov}(\tilde{x}_1{}' + \tilde{x}_2{}', \tilde{y}') - \mathsf{cov}(\tilde{x}_1 + \tilde{x}_2, \tilde{y}).$$

The three terms are not of the same size. If N is large, the change in variance caused by the Pigou–Dalton transfer Δ_{PD} is bounded, whereas Δ_{OA} is not. Specifically, for any $p_1 > p_2$ and any $\delta \in [0, (p_1 - p_2)/2]$,

$$1 \geq \Delta_{\mathrm{PD}} \geq -2. \tag{A.8}$$

The proof of (A.8) is straightforward and thus details are omitted.[12] To compute Δ_{OA}, we use the two extreme cases identified in Proposition 5, in which $\tilde{y}'$ is a sum of Bernoulli variables with the respective probabilities p_j for $j = 3, \ldots, N$. Let $\mu_{\mathrm{OA}} := p_3 + \cdots + p_N \in [Z, Z+1[$ define the expected number of fatalities among the agents not involved in the transfer (with Z being an integer). Then

$$m_y \leq \mathsf{var}(\tilde{y}') \leq M_y,$$

where the definitions of minimum variance $m_y := (1 - \mu_{\mathrm{OA}} + Z)(\mu_{\mathrm{OA}} - Z)$ and maximum variance $M_y := \left(\sqrt{p_3(1-p_3)} + \cdots + \sqrt{p_N(1-p_N)}\right)^2$ hold for both $\mathsf{var}(\tilde{y}')$ and $\mathsf{var}(\tilde{y})$, respectively. Both extreme situations are possible in the sense that there exists a change in the dependence structure of the risks faced by agents $3, \ldots, N$ such that $\mathsf{var}(\tilde{y})$ is either equal to the minimum variance m_y; or equal to the maximum variance M_y. When $N \to \infty$, the maximum variance M_y goes to $+\infty$ and the minimum variance satisfies $m_y \in [0, 1]$ (because $0 \leq \mu_{\mathrm{OA}} - Z < 1$). Thus,

$$m_y - M_y \leq \Delta_{\mathrm{OA}} \leq M_y - m_y,$$

where the lower (upper) bound is obtained when $\tilde{y}$ has maximum (minimum) variance and $\tilde{y}'$ has minimum (maximum) variance. In other words, the lower bound is equal to the maximum decrease in variance because of the change in the dependence structure among the risks $\tilde{x}_3, \ldots, \tilde{x}_N$ and the upper bound is the maximum change of variance because of this change in dependence.

The term Δ_{cov} may increase variability, but what is more important to notice is that the change in the dependence structure of the risks to the agents not involved in the Pigou–Dalton transfer (Δ_{OA}) potentially offsets any other change in variability (Δ_{PD} or Δ_{cov} or their sum), because the effect is unbounded when $N \to \infty$. □

Endnotes

[1] Analogous results were already known in the mathematical statistics literature; see, for example, Hoeffding (1956) and Karlin and Novikoff (1963).

[2] It is, however, not obvious that society should display catastrophe aversion. See Rheinberger and Treich (2016) for an extensive discussion.

[3] We use equiprobable states to ease the exposition of our examples. We emphasize, however, that the results also hold for unequal state probabilities as each state can be broken down into several equiprobable states.

[4] Cox (2012) argues that—unlike income—mortality risk is not fungible and cannot be easily transferred from one individual to another. Regardless of whether one agrees with Cox or not, our definition does not require a direct physical transfer of risk from one individual to another. It merely implies the existence of different policy options under which two individuals face different risks.

[5] For $S := 8$ states $(\omega_1, \ldots, \omega_8)$, it is impossible to find a situation in which the correlation between $\tilde{x}_1'$ and $\tilde{x}_2'$ is equal to 0 (i.e., in which the risks are still independent *after* the risk transfer) and for which the result of Keeney (Proposition 2) would hence hold. It is, however, possible to construct such a situation by invoking more states of the world. We provide one such example in the appendix.

[6] Consider the introduction of a new technology—say a better navigation system in cars. This technology will reduce differences in the individual risk of having a car accident (e.g., by balancing out heterogeneity in driving skills or in car safety features). Therefore, the technology will increase risk equity. On the other hand, it may also raise the dependence of individual accident risks, for example, if there is a software bug that lets the system crash in all the cars at a particular date. Hence, the new technology will increase both risk equity and dependence at the same time. Similar observations have been made, for instance, on systemic risk in the banking industry (see Beale et al. 2011).

[7] This implies that the comparative statics analysis of a change in the risk transfer δ, assuming that the correlation ρ is kept constant, can be fallacious.

[8] Situations (J) and (K) in Section 4.3 provide an example of uncorrelated risks that are still not independent.

[9] Note that the expression of p_M is computed such that the expected number of fatalities is preserved; i.e., $p_M M + (1-p_M)(M+1) = \mu$.

[10] The rearrangement algorithm is based on the following idea. Denote the distribution of fatalities aggregated over all but one agent, say agent i, by $\tilde{d}_{-i} := \sum_{j\neq i} \tilde{x}_j$. The distribution of fatalities becomes less variable iff the correlation ρ_i between $\tilde{d}_{-i}$ and $\tilde{x}_i$ decreases (for any agent $i = 1, 2, \ldots, N$). This result follows from the fact that the variability of the distribution of fatalities can be expressed as $\mathrm{var}(\tilde{d}) = \mathrm{var}(\tilde{x}_i + \tilde{d}_{-i}) = \mathrm{var}(\tilde{x}_i) + \mathrm{var}(\tilde{d}_{-i}) + 2\rho_i\sqrt{\mathrm{var}(\tilde{d}_{-i})}\sqrt{\mathrm{var}(\tilde{x}_i)}$, in which only ρ_i is affected.

[11] In settings with more than two risks, zero correlation does not necessarily imply independence (see situations (J) and (K) in Section 4.3). Yet for two Bernoulli random variables zero correlation always implies independence.

[12] Equation (A.8) follows directly from Lemma 1. There are two possible ways to compute Δ_{PD}: (i) as the difference between the minimum and maximum variances between the two initial risks $\tilde{x}_1$ and $\tilde{x}_2$; or (ii) as the difference between the minimum and maximum variances between the two risks $\tilde{x}_1'$ and $\tilde{x}_2'$ after the Pigou–Dalton transfer. When $p_1 + p_2 > 1$ then the difference between the maximum variance after the transfer and the minimum variance before the transfer is equal to $2(\delta + (1-p_1))$, which is less than unity under the assumption on the range of δ. The difference between the maximum variance before the transfer and the minimum variance after the transfer is equal to $-2(1-p_1)$, which is larger than -2. In the case of $p_1 + p_2 \leq 1$ the bounds are $2(\delta + p_2)$ and $-2p_2$, respectively, and the same conclusion holds.

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