The impact of taxation on optimal portfolio rules of pension funds

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Abstract
The topic of the effects of taxation on optimal portfolio rules, which bridges the gap between public and financial economics, has attracted a sustained attention since the seminal contribution by Domar and Musgrave (1944). Yet analyses of this issue in the case of pension funds are rather limited. In practice, the taxation of pension funds can be applied at three levels: contributions, the investment gain or pensions. The most frequent configuration in industrialized countries is to tax pensions only. By adapting the pension fund asset allocation model of Romaniuk (2007), we show that, when taxing the investment gain or pensions, the participant’s optimal policy becomes riskier in most of the cases, as the speculative fund absolute value increases. The taxation-induced distortions are far less pronounced when taxing contributions, as the only impact on optimal portfolio rules concerns the contribution-hedge fund. We thus emphasize that the current practice of taxing pensions only is not appropriate when trying to reduce the distortions of the taxation system on the portfolio behavior of the investor, and that a taxation applied on contributions would be more adapted.

Keywords: taxation; pension funds; optimal asset allocation; stochastic dynamic programming.

JEL Classification: C61; G11; G23; H22; H39.

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1 Introduction

The analysis of the impact of income taxation on investment begins with the seminal paper by Domar and Musgrave (1944). The authors emphasize that this impact differs depending on whether the investor has the ability to offset portfolio losses against other income or not. In the full offset case, the introduction of taxation enhances the investor’s risk-taking. When no offset is possible, the taxation effect is indeterminate. The problem is later re-examined in the expected utility framework by Mossin (1968), who confirms Domar and Musgrave’s conclusions. A further re-examination by Stiglitz (1969) leads to less clear-cut conclusions, which state that capital gains provisions do not always encourage risk-taking. The effects of wealth, income and capital gains taxation on risk-taking crucially depend on the precise characteristics of the risk aversion of the investor.

The seminal contribution by Constantinides (1983) analyzes the effect of capital gains taxes on optimal consumption and investment behavior in Merton (1971)’s setting. Constantinides (1983) proves a separation theorem of the liquidation policy from the consumption and investment decision.¹ The author emphasizes that the optimal liquidation policy is to realize losses immediately and to defer gains until a forced liquidation occurs. Leland (1999) deals with a portfolio strategy characterized by exogenously fixed portfolio proportions, this configuration leading to the existence, at the optimum, of a no-trade region about the target stock proportions. The author proves that the introduction of capital gains taxes extends the no-trade region. The selling of a stock occurs only once its proportion has substantially exceeded its target level. Leland (1999) emphasizes that the conclusion of Constantinides (1983) - stating that agents should never realize capital gains - does not hold if synthetic selling strategies are not possible: Some capital gains will then be realized.

Dammon et al. (2001a) focus on the consequences of a short-selling restriction, whose presence

¹ The liquidation policy issue emerges because capital gains taxes are due only when the asset is sold.
leads to the invalidation of Constantinides (1983)’s separation theorem. As a result, a trade-off emerges between the diversification motive and the tax costs of trading. The authors conclude that the introduction of capital gains taxes leads to an increased proportion of equity, as capital gains benefit from a favorable tax treatment, when compared to interest income. This one-risky asset analysis is extended by Dammon et al. (2001b) to a world with multiple risky assets, whose existence enhances the diversification motive, and thus the incentive to realize capital gains. Gallmeyer et al. (2006) also consider a setting with capital gains taxes, multiple risky assets and a shorting-the-box restriction (i.e. taking an offsetting position in an identical security is prohibited), while costly shorting (in a different security) is still allowed. They describe a novel trading flexibility strategy, where the investor optimally shorts one of the stocks, with the objective of minimizing the future tax costs of trading. DeMiguel and Uppal (2005) focus on the tax basis: Instead of using the popular approach of taking the weighted average purchase price, they define the exact tax basis. They conclude that the stock proportion substantially increases when introducing capital gains taxes.

The empirical side of the issue of the impact of taxation on the demand of financial assets is first addressed, for the case of the United States, in the seminal contribution by Feldstein (1976), subsequently followed by Hubbard (1985), King and Leape (1998) and Poterba and Samwick (2002). As to other countries, Agell and Edin (1990) deal with the case of Sweden, Hochguertel et al. (1997) analyze the situation in the Netherlands, while Jappelli and Pistaferri (2003) focus on the Italian case. In general, the results state that taxes have an important effect on the demand of portfolio assets, except in Jappelli and Pistaferri (2003), who analyze the rather specific case of Italian life insurance contracts. A positive link between the tax rate and stock holdings is found in Feldstein (1976), Hubbard (1985) and Poterba and Samwick (2002). Agell and Edin (1990) and King and Leape (1998) conclude that the impact of taxes is larger on the probability of investing in an asset than on its portfolio proportion, conditional upon a positive decision on ownership.

The possibility to invest in taxable accounts or tax-deferred (retirement) accounts has recently
led to a growing literature, which analyzes not only the asset allocation decision (choice of each asset portfolio proportion), but also the asset location one (choice of the assets to hold in the taxable and tax-deferred accounts). Dammon et al. (2004), by extending their earlier contribution (Dammon et al., 2001a) in incorporating a tax-deferred account, prove a pronounced preference for holding taxable bonds in the tax-deferred account, and equity in the taxable account (as taxable bonds are more heavily taxed than equity). Yet it appears that this theoretical advice is not followed in practice. In an empirical study of the United States portfolio holdings, Bergstresser and Poterba (2004) find that the portfolio proportions in both taxable and tax-deferred accounts are comparable: In each location, equity amounts to approximately 70% of total assets.

The other theoretical papers provide arguments trying to resolve this apparent dissonance between theory and practice. Shoven and Sialm (2003) introduce tax-exempt bonds and divide stock portfolios in two classes: tax-efficient (represented by passively-managed mutual funds) and tax-inefficient (represented by actively-managed mutual funds). They argue that the tax-deferred account should contain taxable bonds and tax-inefficient stock portfolios, while the taxable account - tax-exempt bonds and tax-efficient stock portfolios. Amromin (2003) emphasizes that, by taking jointly account of the uninsurable labor income risk and imperfect liquidity of tax-deferred accounts, one proves that the precautionary savings motive leads to the presence of bonds in the taxable account. Garlappi and Huang (2006) show that, when considering portfolio constraints (limits on borrowing or short-selling), a preference for investing in stocks in the tax-deferred account can emerge, because of the desire to smooth the future tax subsidy. Trying to bridge the gap between academics, whose main focus is on the interaction between taxable and tax-deferred portfolio decisions, and practitioners, who rather advocate the allocation and location problems separability, Huang (2006) defines the conditions under which this separability holds.

Regarding the issue of the impact of taxation on the pension fund asset allocation, the literature appears as rather limited. The seminal papers by Black (1980) and Tepper (1981), taking an
The integrated view of the firm and its defined benefit pension fund, use the tax-arbitrage argument to advocate an all-bond pension fund portfolio. Yet Bicksler and Chen (1985), by taking account of the joint effects of insurance and taxes in a capital market characterized by imperfections, show that a mixed portfolio can be optimal. Ippolito (1990) also argues that a combination of bonds and equity would be better suited, given the rules for pension funding. Furthermore, as the Tax Reform Act of 1986 led to a reduced incentive for investing pension portfolios in bonds, a substantial stockholding became advisable. Frank (2002) nonetheless empirically shows the positive relation between tax benefits and the bond proportion in pension assets. He stresses that the tax arbitrage opportunities available to firms are not fully used in practice.

One thus concludes that the literature should be extended by a theoretical examination of the impact of taxation on optimal portfolio rules of pension funds. This will constitute this paper’s objective.

Dealing with this issue first requires understanding the characteristics of the pension fund taxation systems. Taxation can be applied on three levels: contributions, the capital accumulation and pensions, which in fact correspond to the stages of entering the fund, accumulating assets and leaving (Charpentier, 1997). Lavigne and Pardo (2000) propose an international comparison of the taxation systems applicable to pension funds in some developed countries. They conclude that taxing pensions only appears as the dominant fiscal structure. The EET system (where T stands for taxation and E for exemption) is used by Canada, Germany, Ireland, Switzerland, the Netherlands, the United Kingdom, the United States. Sweden applies the ETT system, Australia and Japan propose the TTT. As to the Spanish fiscal regime, taxes apply on employee and employer contributions, as well as on pensions; In Italy, pensions are taxed, as well as the pension fund investment returns (Tamburi and Chassard, 2002).

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2 This integration is empirically confirmed by Bodie et al. (1985) and Gallo and Lockwood (1995).
3 In external funds, employer contributions are nonetheless taxed in Germany.
4 In Japan, employee contributions in external funds are nonetheless tax exempt.
It must be stressed that the tax system classification just proposed covers some specificities: For example, in the Netherlands, the employee and employer contributions are both taxed if the pension plan does not benefit from the legal approval; In the United Kingdom, when the legal approval condition is not met, the pension fund fiscal regime is far less generous than the EET benchmark; In Japan, investment returns are tax-exempt, and it is the balance (i.e. pensions less contributions) of some pension plans which is subject to taxation. In the United States, some different taxation configurations are also possible: While the 401(k)/403(b) retirement accounts, or Keogh plans, are EET-taxed (taxes on contributions and on interests and capital gains are deferred and taxed only upon withdrawal), the Roth type retirement account is TEE-taxed (i.e. on contributions only) (Huang, 2006).

When comparing the practice of the pension fund taxation with the literature review just proposed, one first observes that the existing literature focuses on the effects of capital gains taxation only, while pension funds can also be taxed on contributions and pensions. It thus appears as crucial to analyze the impact of taxing the final payoff and the contributions. In fact, one must emphasize that the determining of the effect of the pension taxation is of the highest importance in the case of pension funds - higher than of the definition of the impact of the contribution or capital gains one - as the international comparison of the pension taxation systems showed that taxing pensions is common practice, while taxing contributions or capital gains proves to be relatively rare. One also notes that the literature on the pension fund taxation, which is rather scarce, is focused on defined benefit funds and, even more importantly, takes an integrated view of the firm and its pension fund to analyze the tax arbitrage decision. Yet the actual trend is to separate the two entities in practice, in order to provide a maximum security to the pension fund participants. And, more evidently, the analysis of the defined benefit fund case must be extended by studying the case of defined contribution funds, and possibly of some hybrid ones, like for example the targeted money purchase funds.

Our analysis fills these gaps by proposing a pension fund asset allocation model incorporating
three forms of taxation - on contributions, the investment gain and pensions. By doing so, it allows not only studying separately the effect of each form, yet also comparing these effects, and thus building a classification of the distortion importance induced by each taxation sort. The model also deals with different types of pension funds: defined benefit (DB), defined contribution (DC) and targeted money purchase (TMP). It opts for a separate view of the pension fund and any other entity: The pension fund manager conducts the policy that is optimal from the participant’s point of view.

The pension fund asset allocation model proposed builds on the contribution by Romaniuk (2007). While the previous articles dealing with the pension fund optimal investment strategy issue in Merton (1971)’s and Cox and Huang (1989, 1991)’s settings studied DB (Sundaesran and Zapatero, 1997; Rudolf and Ziemba, 2004) or DC plans (Boulier et al., 2001; Deelstra et al., 2003; Menoncin and Scaillet, 2003; Battocchio and Menoncin, 2004; Cairns et al., 2006; Battocchio et al., 2007), Romaniuk (2007) defined a unified framework in Merton (1971)’s setting for analyzing and comparing the optimal asset allocation policies of the main types of pension funds. This contribution is here extended by the introduction of taxes. It turns out that the presence of contribution, the investment gain and/or pension taxation modifies the optimal asset allocation. The contribution taxation impacts on the contribution-hedge term only, whose absolute value decreases. The investment gain and pension taxation most frequently leads to an increase in the speculative term absolute value, as it becomes divided by one less the tax rate: The riskiness of the investor’s strategy increases. One thus concludes that the current practice of taxing pensions does not appear as the most appropriate when trying to reduce the taxation-induced distortions on portfolio rules, and that taxing contributions would be more adequate.

The paper is organized as follows. The second section presents the optimization programs of DC, DB and TMP funds, with a particular focus on the impact of the taxation system characteristics. The description of the model follows. The optimization program solution is derived and interpreted in the fourth section. The last section concludes.
2 The optimization programs of the main types of pension funds

2.1 The basic setting

Two types of guarantees can be proposed in the case of pension funds. When the guarantee is offered by the pension fund itself (i.e. the fund has to generate a minimum pension), the internal guarantee case applies. When some institution independent from the fund guarantees the payment of a minimum pension, an external guarantee emerges.

Three kinds of pension plans are analyzed: DC, DB and TMP. The DC plan proposes a pension corresponding to the fund asset value \( A \) at the time of retirement \( T \). The DB plan offers a pension equal to the retirement value of the fund liabilities \( L \). The pension of the TMP scheme represents the maximum of the DC pension and the targeted one.

For a DC fund, three subcases are considered: without guarantee and with an internal or external guarantee. Both guarantees are denoted by \( G \).

In the case of a DB fund, the subcases without guarantee or with a guarantee are dealt with. The guarantee \( G \) possibly given takes the form of the fund liabilities \( L \). One introduces the possibility of a rule of surplus sharing between the employee and the employer, which determines the proportion \( \alpha \) of the generated surplus attributable to the participant and the proportion \( 1 - \alpha \) owned by the firm, \( \alpha \) being defined as a constant satisfying \( 0 < \alpha < 1 \).

The optimization programs include three constraints: The utility function argument must be strictly positive at the initial date; It must be non-negative at the retirement date; The variable dynamics constitute the last constraint.

We will assume that, when defining the investment strategy to follow, the fund manager takes the participant’s point of view. As a consequence, the manager and the participant can be seen as a single agent. Depending on whether an internal or external guarantee applies, the optimization program of the fund manager, on a date \( t \) preceding the retirement date \( T \), can take the following forms:
MaxE_t[U(A(T) − G(T))] \quad (1)

or

MaxE_t[U(G(T) + \alpha Max(A(T) − G(T), 0))] \quad (2)

where \( E_t \) denotes the expectation, conditional on the information available in \( t \), \( U \) the utility function and \( 0 < \alpha \leq 1 \).

When \( \alpha = 1 \), the two programs define the general cases of an internal and an external guarantee respectively. The case \( 0 < \alpha < 1 \) characterizes the presence of a rule of surplus sharing.

Program (1) applies for numerous configurations: a DC fund with an internal guarantee, a DB fund without guarantee and with a surplus sharing rule, when \( G(T) = L(T) \), a DB fund with a guarantee and without a surplus sharing rule, with \( G(T) = L(T) \), and a TMP fund targeting the value \( G(T) \) at the date of retirement.

Program (2) corresponds to a DC fund with an external guarantee, when \( \alpha = 1 \), and to a DB fund with a guarantee and a surplus sharing rule, when \( 0 < \alpha < 1 \) and \( G(T) = L(T) \).

### 2.2 Introducing the investment gain and pension taxation

Let us now assume that pensions and/or the investment gain are subject to taxation, at the constant rates \( \tau_1 \) and \( \tau_2 \) respectively, with \( 0 \leq \tau_1, \tau_2 < 1 \).

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5 The program writes \( MaxE_t[U(A(T))] \) under the solvency constraint \( A(T) \geq G(T) \). Following Basak (2002), the program (1) formulation can be adopted. The optimal policy will then automatically respect the solvency constraint if \( \lim_{A \to G^+} U'(A - G) = +\infty \) (Karatzas et al., 1986; Merton, 1990). We impose this condition on the utility function.

6 The participant’s program writes \( MaxE_t[U(L(T) + \alpha Max(A(T) − L(T), 0))] \) under the solvency constraint \( A(T) \geq L(T) \), with \( 0 < \alpha < 1 \). The participant’s main purpose is then the generation of a surplus: This will guarantee that he eventually receives at least the liability value, which will be augmented by a proportion of the generated surplus. Program (1) thus applies.

7 The participant knows that he will receive exactly the liability value at the retirement date, independently of the fund asset value. As a consequence, he is not preoccupied about the adopted strategy. The firm will then take the responsibility of defining the fund policy and maximize \( E_t[U(A(T) − L(T))] \).

8 The optimization program of a TMP plan participant writes \( MaxE_t[U(Max(A(T), G(T)))] \) under the solvency constraint \( A(T) \geq G(T) \). Because of this solvency constraint, the participant will always obtain \( A(T) \). Once again, the generation of a surplus defines the participant’s main purpose: When a surplus is generated, the constraint is met and the amount received \( A(T) \) exceeds the minimum \( G(T) \). The program thus evolves to the formulation (1).
We first deal with pension taxation, when an internal guarantee applies. The DC fund participant then receives \((1 - \tau_1)A(T)\) at the retirement date. One easily sees that his program becomes \(\text{Max}E_t[U((1 - \tau_1)A(T) - G(T))]\). The situation is quite different when turning to a DB fund. The DB fund participant generally obtains \((1 - \tau_1)L(T)\) at the final date, yet his program does not evolve to \(\text{Max}E_t[U(A(T) - (1 - \tau_1)L(T))]\) because of the solvency constraint \(A(T) \geq L(T)\). If the program \(\text{Max}E_t[U(A(T) - (1 - \tau_1)L(T))]\) were used, one would be certain to meet \(A(T) \geq (1 - \tau_1)L(T)\) only. As a consequence, the DB fund program remains \(\text{Max}E_t[U(A(T) - L(T))]\). The TMP fund participant in fact receives \(A(T)\) at the final date, because of the solvency constraint \(A(T) \geq G(T)\), so that the tax applies on the amount \(A(T)\). He will thus maximize \(E_t[U((1 - \tau_1)A(T) - G(T))]\).

Turning to the external guarantee case, one must first emphasize that the tax is paid on the final amount received by the participant, which writes \(G(T) + \alpha\text{Max}(A(T) - G(T), 0)\). As no solvency constraint applies, the optimization program is simply defined as \(\text{Max}E_t[U((1 - \tau_1)(G(T) + \alpha\text{Max}(A(T) - G(T), 0))]\).

Let us now focus on the investment gain taxation characteristics. It is assumed that the tax is paid at a rate \(\tau_2\) on the final investment gain of the participant, which is defined as the pension amount received at retirement less the initial asset value \(A(0)\). In the internal guarantee case, the DC fund participant then uses the program: \(\text{Max}E_t[U(A(T) - \tau_2(A(T) - A(0)) - G(T))\), which is equivalent to \(\text{Max}E_t[U((1 - \tau_2)A(T) + \tau_2A(0) - G(T))]\). As to the DB fund,

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9 When surplus sharing applies, the DB fund participant receives \((1 - \tau_1)(L(T) + \alpha\text{Max}(A(T) - L(T), 0))\) at the final date. Because of the constraint \(A(T) \geq L(T)\), the program \(\text{Max}E_t[U(A(T) - L(T))]\) is still valid.

10 If we denote by \(PB\) the pension benefit paid to the participant, the investment gain tax will amount to \(\tau_2[PB(T) - A(0)]\). Another possible modelling of the investment gain tax would be \(\tau_2[A(T) - A(0)]\). To compare both possibilities, let us first note that \(PB(T) = A(T) + (PB(T) - A(T))\). In our modelling, the participant thus pays a tax on the amount \(A(T)\), to which is added a tax on \(PB(T) - A(T)\). This second tax element will constitute a tax reduction if \(PB(T) < A(T)\) and a tax surplus if \(PB(T) > A(T)\). The fiscal authority thus imposes a payment on the effective pension amount received by the retiree. The tax is thus higher if the pension benefit exceeds the final asset value of the pension fund, and lower in the opposite case.

11 The constraint \(A(T) \geq G(T)\) is then met provided that \(A(T) \geq A(0)\). As the event \(A(T) < A(0)\) appears as highly unlikely (even if theoretically possible), we will use the proposed program formulation.

12 In the case of a continuous-time investment gains (and losses) taxation process, the investor would continuously pay a tax of \(\tau_2 dA\), which would also result in the utility function argument just defined.
If the program \( \text{Max} E_t[U(A(T) - (L(T) - \tau_2(L(T) - A(0))))] \), equivalent to \( \text{Max} E_t[U(A(T) - (1 - \tau_2)L(T) - \tau_2 A(0))] \), were used, the solvency constraint \( A(T) \geq L(T) \) would not always have been met at the final date, as only meeting \( A(T) \geq L(T) - \tau_2(L(T) - A(0)) \) would have been guaranteed. The initial program form (characteristic of a no-taxation world) \( \text{Max} E_t[U(A(T) - L(T))] \) thus remains valid.\(^{13}\) The TMP fund participant maximizes \( E_t[U((1 - \tau_2)A(T) + \tau_2 A(0) - G(T))] \): Because of the constraint \( A(T) \geq G(T) \), he always receives \( A(T) \) at the retirement date, so that the investment gain taxation applies on \( A(T) - A(0) \).

Regarding the external guarantee case, introducing the investment gain taxation gives the following optimization program:

\[
\text{Max} E_t[U(G(T) + \alpha \text{Max}(A(T) - G(T), 0) - \tau_2 [G(T) + \alpha \text{Max}(A(T) - G(T), 0) - A(0))], which can be rewritten in the form: \( \text{Max} E_t[U((1 - \tau_2) [G(T) + \alpha \text{Max}(A(T) - G(T), 0)] + \tau_2 A(0))] \).

When analyzing the investment gain taxation issue, one has to define the loss offset ability of the investor. When the investor has the ability to offset losses against other income, the Treasury shares not only in the investor’s gains, but also in his losses (Domar and Musgrave, 1944). In our model, we assume a full loss offset possibility. Let us take the example of the internal guarantee, DC fund participant case, to analyze the no offset configuration. The participant would then maximize \( E_t[U(A(T) - \tau_2 \text{Max}(A(T) - A(0); 0) - G(T))] \): An option thus appears in the utility function argument. If a partial offset is permitted, the agent would maximize \( E_t[U(A(T) - \tau_2 \text{Max}(A(T) - A(0); 0) + \beta \tau_2 \text{Max}(A(0) - A(T); 0) - G(T))] \), with \( \beta \) the proportion being allowed for offsetting. We will not consider the no offset or partial offset cases, as our modelling of the investment gain taxation renders the event of capital losses (which would mathematically mean \( A(T) < A(0) \)) unlikely.

Summarizing all the above mentioned possibilities, one concludes that the presence of pension and/or the investment gain taxation leads to the following optimization program configurations:

\(^{13}\) In the case of surplus sharing, the DB fund participant pays a tax amounting to \( \tau_2[L(T) + \alpha \text{Max}(A(T) - L(T), 0) - A(0)] \) (instead of \( \tau_2[L(T) - A(0)] \) as in the benchmark case). Yet the program is still \( \text{Max} E_t[U(A(T) - L(T))] \), as the constraint \( A(T) \geq L(T) \) has to be met.
The internal guarantee case is characterized by $\text{MaxE}_t[U((1 - \tau_1 - \tau_2)A(T) + \tau_2A(0) - G(T))]$ for the DC and TMP fund participants, while the DB fund program is identical to the one of the no-taxation world: $\text{MaxE}_t[U(A(T) - G(T))]$; The external guarantee case yields the program $\text{MaxE}_t[U((1 - \tau_1 - \tau_2)[G(T) + \alpha\text{Max}(A(T) - G(T), 0)] + \tau_2A(0))]$.

## 3 The model

It is assumed that the financial market characteristics meet the usual assumptions, in particular the completeness one.

$N$ risky assets are available in the economy. Their dynamics obey:

\[ dS_t = I_S \mu_S(t, Z(t))dt + I_S \sigma_S(t, Z(t))dB(t) \tag{3} \]

where $I_S$ denotes an $(N \times N)$ diagonal matrix valued function of $S(t)$, $\mu_S(t, Z(t))$ an $(N \times 1)$-dimensional vector, $\sigma_S(t, Z(t))$ an $(N \times M)$ matrix valued function, $B(t)$ an $(M \times 1)$-dimensional Wiener process in $R^M$, $Z(t)$ a $(K \times 1)$-dimensional vector of state variables.

The price $S_i(t)$ of the $i$th risky asset then follows the stochastic differential equation (SDE):

\[ dS_i(t) = S_i(t)\mu_{S_i}(t, Z(t))dt + S_i(t)\sigma_{S_i}(t, Z(t))dB(t) \tag{4} \]

with $\mu_{S_i}(t, Z(t))$ defined as a bounded function of $t$ and $Z$, $\sigma_{S_i}(t, Z(t))$ a bounded $(1 \times M)$ vector valued function of $t$ and $Z$.

The $K$ stochastic state variables have the following dynamics:

\[ dZ(t) = I_Z \mu_Z(t, Z(t))dt + I_Z \sigma_Z(t, Z(t))dB(t) \tag{5} \]

where $I_Z$ stands for a $(K \times K)$ diagonal matrix valued function of $Z(t)$, $\mu_Z(t, Z(t))$ a bounded

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14 Let us note that if pensions are not taxed, $\tau_1 = 0$, while when $\tau_2 = 0$, no taxation on the investment gain applies.

15 For the financial market to be complete, we will assume $N = M$. 
A riskless asset is also available. Its price $\eta(t)$ evolves according to the dynamics:

$$d\eta(t) = \eta(t)r(t, Z(t))dt$$

with $r(t, Z(t))$ the instantaneously riskless interest rate, which depends on the state variables $Z$.

Contributions are brought continuously into the fund. They are defined as a constant proportion of the employee’s salary. The employee contributes a proportion $\zeta_1$, while the employer a proportion $\zeta_2$, with $\zeta_1, \zeta_2 \geq 0$. The participant’s contributions are taxed at a constant rate $t_1$, while the firm’s ones at $t_2$, with $0 \leq t_1, t_2 < 1$.

The wage $Y$ dynamics take the form of the following SDE:

$$dY(t) = Y(t)\mu_Y(t, Z(t))dt + Y(t)\sigma_Y(t, Z(t))dB(t)$$

where $\mu_Y(t, Z(t))$ denotes a bounded function of $t$ and $Z$, $\sigma_Y(t, Z(t))$ a bounded $(1 \times M)$ vector valued function of $t$ and $Z$.

A minimum guarantee $G$ can be offered by or to a pension fund. Its dynamics obeys:

$$dG(t) = G(t)\mu_G(t, Z(t))dt + G(t)\sigma_G(t, Z(t))dB(t)$$

where $\mu_G(t, Z(t))$ stands for a bounded function of $t$ and $Z$, $\sigma_G(t, Z(t))$ a bounded $(1 \times M)$ vector valued function of $t$ and $Z$.

The utility function of the participant depends on a variable $X$, which is defined by $X(T) = (1 - \ldots$

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16 The proposed modelling allows to represent various configurations: Only the employee contributes ($\zeta_1 > 0$, $\zeta_2 = 0$), only the employer contributes ($\zeta_1 = 0$, $\zeta_2 > 0$) or both contribute ($\zeta_1 > 0$, $\zeta_2 > 0$). Regarding taxation: Only the participant’s contributions are taxed ($t_1 > 0$, $t_2 = 0$), only the firm’s contributions are taxed ($t_1 = 0$, $t_2 > 0$), both are taxed ($t_1 > 0$, $t_2 > 0$), at identical rates ($t_1 = t_2$) or at different rates ($t_1 \neq t_2$), or none of the contributions are subject to taxation ($t_1 = t_2 = 0$).
\( \tau_1 - \tau_2 ) A(T) + \tau_2 A(0) - G(T) \) for DC and TMP fund participants, \( X(T) = A(T) - G(T) \) for a DB fund participant in the internal guarantee setting, and

\[
X(T) = (1 - \tau_1 - \tau_2) [G(T) + \alpha \max(A(T) - G(T), 0)] + \tau_2 A(0), \quad \text{with } 0 < \alpha \leq 1,
\]

in the external guarantee case. The utility function thus takes the terminal form \( U(X(T)) \). It is assumed to be increasing and concave in \( X \) and to respect the Inada conditions: \( \lim_{X \to \infty} U_X = 0 \) and \( \lim_{X \to 0} U_X = \infty \), with \( U_X \) the derivative of \( U \) with respect to \( X \).

The optimization program of the manager is then defined by:

\[
\max_{xS} E_t [U(X(T))] \tag{9}
\]

s.c.d \( X(t) = X(t) \mu_X(t, x_S, X(t), Z(t)) dt + X(t) \sigma_X(t, x_S, X(t), Z(t)) dB(t) \)

\( dZ(t) = I_Z \mu_Z(t, Z(t)) dt + I_Z \sigma_Z(t, Z(t)) dB(t) \)

\( X(0) > 0 \)

\( X(T) \geq 0 \)

where \( xS \) denotes an \((N \times 1)\) vector of the proportions of \( X \) invested in the risky assets, the variable \( X \) dynamics being written in a general form, with \( \mu_X(t, x_S, X(t), Z(t)) \) a function of \( t, x_S, X \) and \( Z \) and \( \sigma_X(t, x_S, X(t), Z(t)) \) a \((1 \times M)\) vector valued function of \( t, x_S, X \) and \( Z \).

4 Definition of the optimal policy

4.1 The internal guarantee case

The fund assets \( A \) are invested in the risky assets \( S_i \) \((i = 1, 2, ..., N)\) and in the riskless asset \( \eta \). \( X_{S_i} \) denotes the number of asset \( S_i \) and \( X_{\eta} \) the amount of riskless asset held in the portfolio.

When an internal guarantee applies, \( X(T) = (1 - \tau_1 - \tau_2) A(T) + \tau_2 A(0) - G(T), \) which yields:

\[
X(T) = (1 - \tau_1 - \tau_2) \left[ \sum_{i=1}^{N} X_{S_i}(T) S_i(T) + X_{\eta}(T) \eta(T) \right] + \tau_2 A(0) - G(T) \tag{10}
\]
Relative to $X$, the risky asset $S_i$ is held in the proportion $x_{S_i} \equiv \frac{X S_i}{X}$, the riskless asset $\eta$ in the proportion $x_{\eta} \equiv \frac{X \eta}{X}$ and the guarantee $G$ (negatively) in the proportion $x_{G} \equiv \frac{G}{X}$. The following identity: 

$$
(1-\tau_1-\tau_2) \left[ \sum_{i=1}^{N} x_{S_i} + x_{\eta} \right] + \frac{\tau \epsilon_A(0)}{X} - x_{G} = 1
$$

is met. The variable $X$ dynamics are then defined by:

$$
\frac{dX}{X} = (1-\tau_1-\tau_2) \left[ \sum_{i=1}^{N} \frac{x_{S_i}}{S_i} + x_{\eta} \frac{d\eta}{\eta} + \left( (1-t_1) \zeta_1 + (1-t_2) \zeta_2 \right) \frac{Y}{X} dY \right] - x_{G} \frac{dG}{G}
$$

with $\left( (1-t_1) \zeta_1 + (1-t_2) \zeta_2 \right) \frac{Y}{X} dY$ appearing because of the continuous accumulation of the contribution process.

When one replaces the dynamics (4), (6), (7) and (8) and uses the identity

$$
(1-\tau_1-\tau_2) \left[ \sum_{i=1}^{N} x_{S_i} + x_{\eta} \right] + \frac{\tau \epsilon_A(0)}{X} - x_{G} = 1,
$$

the following dynamics are obtained:

$$
\frac{dX}{X} = \left[ (1-\tau_1-\tau_2) \left( x_S' (\mu_S - r 1_N) + r \frac{1-\tau \epsilon_A(0)}{1-\tau_1-\tau_2} + \left( (1-t_1) \zeta_1 + (1-t_2) \zeta_2 \right) \frac{\mu}{X} \frac{Y}{Y} \right) \right] dt + \left[ (1-\tau_1-\tau_2) \left( x_S' \sigma_S + \left( (1-t_1) \zeta_1 + (1-t_2) \zeta_2 \right) \frac{\sigma}{X} \frac{Y}{Y} \right) \right] dB
$$

the prime ' standing for a transpose.

Let us define the indirect utility function $J$, increasing and strictly concave in $X$, by:

$$
J(t, X(t), Z(t)) \equiv \max_{x_0} E_t[U(X(T))]
$$

The Hamilton-Jacobi-Bellman (HJB) equation then takes the form:

$$
\max_{x_0} DJ = 0
$$
with $D$ the Dynkin operator.

By deriving the Dynkin of $J$ with respect to $x_S$, one obtains the system of first order conditions:

\[
0_N = (\mu_S - r1_N)(1 - \tau_1 - \tau_2)J_XX
\]

\[
+ \begin{bmatrix}
\Sigma_{SS}xs(1 - \tau_1 - \tau_2)^2 \\
+\Sigma_{SY} ((1 - t_1)\zeta_1 + (1 - t_2)\zeta_2) \frac{X}{X}(1 - \tau_1 - \tau_2)^2 \\
-\Sigma_{SGxG}(1 - \tau_1 - \tau_2) \\
+\Sigma_{SZ}l'_Z J_{XZ}X(1 - \tau_1 - \tau_2)
\end{bmatrix} J_{XX}X^2
\]

where subscripts on $J$ denote partial derivatives and $\Sigma_{ij} \equiv \sigma_i \sigma'_j$ the covariance matrix of the variables $i$ and $j$.

The optimal vector of proportions $x_S$ writes:

\[
x_S = -(\Sigma_{SS})^{-1}(\mu_S - r1_N) \frac{J_X}{J_{XX}X} \frac{1}{1 - \tau_1 - \tau_2} \frac{J_{XX}X}{J_{XX}X} \frac{1}{1 - \tau_1 - \tau_2} \\
-\Sigma_{SY} ((1 - t_1)\zeta_1 + (1 - t_2)\zeta_2) \frac{X}{X} \\
+(\Sigma_{SS})^{-1}\Sigma_{SGxG} \frac{1}{1 - \tau_1 - \tau_2} \\
-(\Sigma_{SS})^{-1}\Sigma_{SZ}l'_Z J_{XZ}X \frac{1}{J_{XX}X} \frac{1}{1 - \tau_1 - \tau_2}
\]

The optimal investment strategy is composed of the four following elements: the speculative fund, the preference independent contribution- and guarantee-hedge portfolios and a preference dependent state variable-hedge component.

The contribution tax influences the contribution-hedge term only, which, instead of taking the form $-(\Sigma_{SS})^{-1}\Sigma_{SY} (\zeta_1 + \zeta_2) \frac{X}{X}$, writes

$-(\Sigma_{SS})^{-1}\Sigma_{SY} ((1 - t_1)\zeta_1 + (1 - t_2)\zeta_2) \frac{X}{X}$. The employee and employer contribution rates become multiplied by one less their respective tax rates. The contribution-hedge term absolute value decreases with the tax rates $t_1$ and $t_2$. An economic interpretation for this weaker importance of contributions in determining the optimal portfolio rule is that, when compared with
the no-taxation case, these contributions now play a less important role in the wealth accumulation, as some portion of them is initially subtracted in the form of taxes. As a consequence, the contribution hedging demand decreases.

The investment gain and pension taxation leads to multiplying all the portfolio terms, except the contribution-hedge one, by \( \frac{1}{1-\tau_1 - \tau_2} \), so that their absolute value increases with \( \tau_1 \) and \( \tau_2 \). These forms of taxation thus render the investor’s policy riskier, when compared with the no-taxation case, as the speculative fund absolute value rises. This is because the introduction of the investment gain or pension taxes diminishes the final payoff of the scheme, as the final asset value becomes multiplied by \( (1 - \tau_1 - \tau_2) \). The investor thus compensates this final payoff decrease by increasing his policy’s riskiness, with the objective of yielding higher returns, and thus maintaining the same utility level as in the no-taxation world.

One thus concludes that the fiscal authority should be cautious when determining the taxation system characteristics. Our results show that taxing the investment gain or pensions can prove to be dangerous, as it increases the riskiness of the participant’s investment behavior; the higher the tax rate, the riskier the asset allocation policy chosen. On the contrary, taxing contributions has a far less pronounced effect, as it only decreases the contribution-hedge fund absolute value.

Let us recall that, in the internal guarantee case, the investment gain and pension taxation leads to the program \( \text{Max}E_t[U((1 - \tau_1 - \tau_2)A(T) + \tau_2A(0) - G(T))] \) for the DC and TMP fund participants, while the DB fund program is identical to the one of the no-taxation world: \( \text{Max}E_t[U(A(T) - G(T))] \). The contribution taxation nonetheless identically applies on DC, DB and TMP funds. As a consequence, the contribution-hedge term is modified for all of the fund types, while the remaining three portfolio terms evolve in the DC and TMP fund cases only. The introduction of the investment gain and pension taxation thus does not enhance the riskiness of the investor’s strategy for the DB fund, in the internal guarantee setting.
4.2 The external guarantee case

The variable $X$ is now defined by:

$$X(T) = (1 - \tau_1 - \tau_2) [G(T) + \alpha \text{Max}(A(T) - G(T), 0)] + \tau_2 A(0),$$

with $0 < \alpha \leq 1$. By denoting by $C$ the call incorporated in $X$, whose payoff at maturity $T$ writes $C(T) = \text{Max}(A(T) - G(T), 0)$, one can reformulate the variable $X$ definition:

$$X(T) = (1 - \tau_1 - \tau_2) [G(T) + \alpha C(T)] + \tau_2 A(0),$$

The variable $X$ is invested in the guarantee $G$ in the proportion $x_G$ and in the call $C$ in the proportion $x_C \equiv \frac{C}{X}$, with the identity $(1 - \tau_1 - \tau_2) [x_G + \alpha x_C] + \frac{\tau_2 A(0)}{X} = 1$. The variable $X$ dynamics then writes:

$$\frac{dX}{X} = (1 - \tau_1 - \tau_2) \left[ x_G \frac{dG}{G} + \alpha x_C \frac{dC}{C} \right]$$

In order to define the dynamics of the call $C(t, A, G, Z_1, Z_2, ..., Z_K)$, one applies Ito's lemma to the function $C$ and uses the non-arbitrage portfolio of Black and Scholes (1973) to finally obtain:

$$dC = \left[ r + \sum_j C_j \frac{j}{C} (\mu_j - r) \right] dt + \sum_j C_j \frac{j}{C} \sigma_j dB$$

(15)

where $j = \{A, G, Z_1, Z_2, ..., Z_K\}$ and subscripts on $C$ denote partial derivatives.

One now needs to derive the variable $A$ dynamics. The composition of $A$ is: $A = \sum_{i=1}^{N} X_{S_i} S_i + X_{\eta} \eta$. The variable $A$ being invested in the risky assets $S_i$ in the proportion $\chi_{S_i} \equiv \frac{X_{S_i}}{A}$ and in $\eta$ in the proportion $\chi_{\eta} \equiv \frac{X_{\eta}}{A}$, the following identity: $\sum_{i=1}^{N} \chi_{S_i} + \chi_{\eta} = 1$ being met, the dynamics of $A$ writes:

$$dA \frac{A}{A} = \sum_{i=1}^{N} \chi_{S_i} dS_i \frac{S_i}{S_i} + \chi_{\eta} \frac{d\eta}{\eta} + ((1 - \tau_1) \zeta_1 + (1 - \tau_2) \zeta_2) \frac{Y}{A} dY$$

---

17 See Romaniuk (2007) for details.
the term \([(1 - t_1) \zeta_1 + (1 - t_2) \zeta_2] \frac{Y}{A} \mu_Y\) being once again introduced because of the contribution continuous accumulation.

One then replaces the dynamics (4), (6) and (7) and uses the identity \[\sum_{i=1}^{N} \chi_{S_i} + \chi_n = 1\] to finally obtain the following dynamics of \(A\):

\[
\frac{dA}{A} = \left[\chi'_{S} \left(\mu_S - r1N\right) + r + ((1 - t_1) \zeta_1 + (1 - t_2) \zeta_2) \frac{Y}{A} \mu_Y\right] dt + \left[\chi'_{S} \sigma_S + ((1 - t_1) \zeta_1 + (1 - t_2) \zeta_2) \frac{Y}{A} \sigma_Y\right] dB
\]  

(16)

The dynamics of the call becomes, by using the equations (15) and (16):

\[
\frac{dC}{C} = \left[\chi'_{S} \left(\mu_S - r1N\right) + ((1 - t_1) \zeta_1 + (1 - t_2) \zeta_2) \frac{Y}{C} \mu_Y\right] dt + \left[\chi'_{S} \sigma_S + ((1 - t_1) \zeta_1 + (1 - t_2) \zeta_2) \frac{Y}{C} \sigma_Y\right] dB
\]  

(17)

Let us recall that \(\frac{dX}{X} = (1 - \tau_1 - \tau_2) \left[\chi_{G} \frac{dG}{G} + \alpha \chi_{C} \frac{dC}{C}\right]\). One then needs to use the definitions of \(x_G\) and \(x_C\) and the identities \((1 - \tau_1 - \tau_2) [\chi_{G} + \alpha \chi_{C}] + \frac{\tau_2\chi(0)}{X} = 1\) and \(\chi_{S_i} = x_{S_i} \frac{S_i}{X}\) to finally obtain, after rearranging terms, the dynamics of the variable \(X\):

\[
\frac{dX}{X} = (1 - \tau_1 - \tau_2) \left[\frac{1 - \frac{\tau_2\chi(0)}{X}}{1 - \tau_1 - \tau_2} + \alpha C_{A} x'_{S} (\mu_S - r1N) \right]
\]

\[
+ \alpha C_{A} ((1 - t_1) \zeta_1 + (1 - t_2) \zeta_2) \frac{Y}{X} \mu_Y
\]

\[
+ (1 + \alpha C_{G}) x_{G} (\mu_G - r) + \alpha \frac{C'_{G} I_{Z} (\mu_Z - r1K)}{X}
\]

\[
+ (1 - \tau_1 - \tau_2) \left[\alpha C_{A} x'_{S} \sigma_S + \alpha C_{A} ((1 - t_1) \zeta_1 + (1 - t_2) \zeta_2) \frac{Y}{X} \sigma_Y\right]
\]

\[
+ (1 + \alpha C_{G}) x_{G} \sigma_G + \alpha \frac{C'_{G} I_{Z} \sigma_Z}{X}
\]

(18)

The system of first order conditions is then easily derived:
\[ 0_N = \left[ (\alpha C_A) \left( \mu_S - r1_N \right) \right] (1 - \tau_1 - \tau_2) J_{XX} X \]
\[ + \left[ (\alpha C_A)^2 \sum_{SS} x_S + \left( (\alpha C_A)^2 \left( (1 - t_1) \zeta_1 + (1 - t_2) \zeta_2 \right) \frac{Y}{X} \right) \sum_{SY} \right] \]
\[ + (\alpha C_A (1 + \alpha C_G) x_G) \sum_{SG} + (\alpha^2 C_A \frac{1}{X}) \sum_{SZ} I_{XZ} C_{Z} \]
\[ (1 - \tau_1 - \tau_2)^2 J_{XX} X^2 \]
\[ + [(\alpha C_A) \sum_{SZ} I_{XZ} X (1 - \tau_1 - \tau_2)] \]

The optimal vector of proportions \( x_S \) then writes:

\[ x_S = - (\sum_{SS})^{-1} \left( \mu_S - r1_N \right) \frac{J_X}{J_{XX} X} \frac{1}{\alpha C_A} \frac{1}{1 - \tau_1 - \tau_2} \]
\[ - (\sum_{SS})^{-1} \sum_{SY} \left( (1 - t_1) \zeta_1 + (1 - t_2) \zeta_2 \right) \frac{Y}{X} \]
\[ - (\sum_{SS})^{-1} \sum_{SG} \left[ \frac{1 + \alpha C_G}{\alpha C_A} \right] \frac{1}{C_A X} \]
\[ - (\sum_{SS})^{-1} \sum_{SZ} I_{XZ} \frac{1}{J_{XX} X} \frac{1}{\alpha C_A} \frac{1}{1 - \tau_1 - \tau_2} \]

The optimal asset allocation policy is defined as the sum of the five following elements: the modified speculative fund, the preference independent contribution-hedge term, the modified preference independent guarantee-hedge fund and two modified state variable-hedge components, one preference independent call-related and the other preference dependent.

Regarding the impact of the contribution taxation, it is identical to the one characteristic of the internal guarantee setting. The contribution-hedge term turns to be exactly like the former one. Once again, as the employee and employer contribution rates become multiplied by one less their respective tax rates, the contribution-hedge fund absolute value decreases with \( t_1 \) and \( t_2 \).

As to the investment gain and pension taxation, it impacts on the modified speculative fund and on the modified preference dependent state variable-hedge component only. Contrary to the internal guarantee case, the modified guarantee-hedge term is not influenced by the introduc-
tion of taxation. The new modified call-related state variable-hedge component is also taxation-independent. When introducing taxation, the modified speculative fund and the modified preference dependent state variable-hedge component become divided by \((1 - \tau_1 - \tau_2)\): Their absolute values thus increase with \(\tau_1\) and \(\tau_2\). As the modified speculative fund absolute value increases when compared with the no-taxation world, the introduction of taxation in the external guarantee setting enhances the riskiness of the investor’s behavior.

When comparing the cases of internal and external guarantees, one concludes that the contribution taxation acts in a similar way in both of the cases: It decreases the contribution-hedge fund absolute value. The investment gain and pension taxation has a comparable effect in the internal and external guarantee settings: With the exception of the case of a DB fund with internal guarantee, it renders riskier the investor’s policy by increasing the speculative fund absolute value.

5 Conclusion

The theoretical examination of the impact of taxation on optimal portfolio rules of defined benefit, defined contribution and targeted money purchase pension funds constitutes this paper’s objective. This is done by extending the pension fund asset allocation model by Romaniuk (2007) to include taxation.

In the pension fund practice, taxation applies on contributions, the investment gain and/or pensions, with taxing pensions only being the most frequent configuration. Yet the literature focuses on determining the impact of capital gains taxes mainly. As a consequence, there exists a gap to be filled: the analysis of the effects on the two other taxation types. The purpose of our analysis is then twofold: the definition of the impact of each taxation sort separately and the comparison of their relative effects. The final objective is to determine the best taxation type(s), understood like the one(s) leading to the least important distortions in the investment policy of the fund participant.
The results prove that the contribution taxation does impact of optimal portfolio rules in a rather limited manner: The only impact is the decrease of the contribution-hedge fund absolute value. The investment gain and pension taxation leads to far more pronounced effects: In particular, it generally leads to an increase in the speculative fund absolute value. The participant’s investment policy thus becomes riskier. One can thus conclude that, contrary to the current pension fund practice which commonly taxes pensions, the purpose of reducing taxation-induced distortions on the pension fund investment policies would rather lead to opt for the contribution taxation.

References


