Alternating Monopoly and Tacit Collusion*

Andrea Amelio† Sara Biancini‡

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Abstract

This paper considers the use of the alternating monopoly strategy (AMS) as a (tacit) collusion device. We show that firms may choose this strategy in particular environments, when other collusive strategies are also feasible. In particular, we stress how the presence of an observable move (entry), distinct from the competitive stage (price setting), can serve as a coordination device, reducing the costs of monitoring in incomplete information environments. The paper thus shows that AMS may be preferable to classic market sharing collusion (MSE) and in some cases is the only collusive equilibrium.

JEL classification: L41.

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Introduction

Under tacit collusion firms are able to collectively exert market power preventing the emergence of the competitive outcome without relying on communication or a formal cartel agreement. The set of schemes that colluding firms can deploy to sustain tacit collusion is broad and has been largely investigated in the literature.¹ However, among all these coordination schemes, very little work has been done on the ability of firms to tacitly share the market on a temporal basis.² This paper provides an analysis of this particular coordination scheme. We consider the possibility

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†GREMAQ, Toulouse University. E-mail: andrea.amelio@univ-tlse1.fr

‡Corresponding author. European University Institute. Address: EUI-Economic Department, via della Piazzola 43, 50133 Firenze (Italy). Tel: +39 055 4685931. E-mail: sara.biancini@eui.eu.

¹For a complete review see Ivaldi, Jullien, Rey, Seabright, and Tirole (2003).

²One exception is the auction literature on bid rigging. For a recent contribution, see Aoyagi (2003).
for firms to share the market on a temporal basis, exerting market power in turns. We refer to this as “alternating monopoly strategy” (hereafter AMS). We find that under some conditions AMS can be a viable alternative to standard price collusion and firms have incentives to adopt this strategy because it improves their profits. The ability to share the market on a temporal basis can be seen as an alternative to the well known and understood ability to share (tacitly or explicitly) the geographical market.\(^3\) In both types of market division, firms are able to extract monopoly rents on their assigned territory (geographic or temporal). There are analogies between splitting the market geographically and enjoying half of the demand; and splitting the market temporally and enjoying the total demand for half of the time. These two territorial allocation rules give firms similar incentives. Although less attention has been devoted to intertemporal market division, we think that this is likely to be an important coordination device in several markets.

Many stylized industry facts seem to suggest firms coordinate temporally.\(^4\) For instance, there is some evidence of behavior that seemingly reproduces time coordination among large firms producing highly substitutable goods. One example, provided in Dixit and Nalebuff (1991), are the Coca-cola and Pepsi discount campaigns. Another possible example are media markets. TV stations, for instance, carefully plan in advance the time schedule for their programs. They often schedule their premium contents (like movies, new TV-series or sports events) in such a way to avoid competing head-to-head for the same audience. To actively compete in the product market is a fundamental choice of the firm and therefore the release of a new product into the market is a strategic decision. This creates scope for a new form of competition or, in the case of collusive equilibria, a new dimension for firms to agree on. Einav (2002) in his extensive study on the U.S. motion picture industry claims that the peculiar characteristics of this market (roughly constant prices and a short product life cycle) make the release date of films the key strategic decision.\(^5\) Moving from the observations of industry behaviors, we analyze some market characteristics which may favor temporal coordination. The first of them is transaction prices unobservability: tacit collusion is in general more difficult when monitoring prices is not possible. However, temporal coordination may be seen as a possible way out in this case, because price observation is less important when firms take turns on the market. In practice, transaction prices are often unobservable. In the music industry, this is due for instance to the discount policies of distributors. In

\(^3\)See for instance Capozza and Van Order (1978) and MacLeod, Norman, and Thisse (1987).

\(^4\)The spatial differentiation literature offers a complementary explanation to the evidence of temporal coordination. See Bester, de Palma, Leininger, Thomas, and von Thadden (1996) In the attempt of differentiation, firms suffers from coordination problem and can solve it by coordinating temporally. Our model allows us to look at collusive issues in a repeated game framework.

\(^5\)As reported in Zillante (2005), a breakfast meeting between Jeffrey Katzenberg, a Dreamworks studio founder, and Harvey Weinstein, co-chairman of Miramax Films supports Einav’s (2000) view. During this meeting, the two agreed to push back the release of one of two films both scheduled for Christmas by five days. This indicates the fine tuning of release dates is not trivial and it can crucially impact on firms’ profits.
a recent EU Commission decision (Sony/BMG Case No COMP/M. 33 33), the difficulty of observing competitors’ prices was explicitly considered as to having reduced the likelihood of collusive behavior. In our paper, we look at situations in which price unobservability may not be as an obstacle to collusion as commonly thought. Another factor which can favor intertemporal coordination is the presence of massive advertising. The release of new products, in particular premium products, is easily observed because they are systematically announced through advertising campaigns. For instance in the media market, movies which represent big events for the Majors, are launched by advertising campaigns which announce the time they will be on the screens with astonishing precision. The typical marketing strategy (big advertising campaign for events in precise periods of the year, the month or the week) offers an observable activity which can serve as a coordination device for firms operating in these sectors. In this sense, adverts are not only an informative or persuasive device, but also a means to exchange information to sustain collusion.\(^{6}\)

The present paper investigates a form of tacit collusion based on AMS. We show how demand uncertainty (and the implied price monitoring problem) can favor the emergence of an alternating monopoly, as opposed to the standard market sharing collusive agreement. Under some conditions, AMS can solve the uncertainty problem by relying on less or cheaper information: competitor’s products availability in the market as opposed to the price or the quantity produced by the competitor. We approach the problem by using a repeated game framework with price unobservability and demand uncertainty. Contrarily to prices, firms’ entry decision are publicly observable. In this context we compare AMS and MSS equilibria. We show that, under some conditions, AMS may dominate MSS and there are cases in which AMS is the only feasible form of tacit collusion.

After illustrating the main literature contributions on the topic, the paper proceeds as follows. Section 1 introduces the framework and characterizes the MSS and AMS equilibria. Section 2 compares the outcome of the two strategies in terms of feasibility and ranks them on a Pareto criterion. Section 3 presents some extensions and robustness checks. Section 4 discusses the results, derives some policy implications and concludes.

**Related literature**

The economic literature has considered, to a limited extent, the release of new products as a strategic variable. The literature on R&D focuses on the fact that reducing the “time to market” is an optimal strategy for all the firms, since it can assure temporary monopoly rent on the innovation. Much less attention has been devoted to the possibility that some firms, with a portfolio of new products, can deliberately choose to delay the introduction of the products to a more favorable period, perhaps

\(^{6}\)This is (in some sense) related to the issue of information sharing between firms as a facilitating device to achieve collusion. See Kühn (2001) for additional information on the subject.
when they are let alone in the market. In the marketing literature, however, this issue has been widely developed. It is recognized that timing decisions play a central role in the strategy of the firms (Krider and Weinberg, 1998 and Ainslie, Drèze, and Zufryden, 2002). This may be the case in markets in which bringing a new good into the market requires some time (for instance to reach the potential customers through advertising) and consumers are willing to buy new goods repeatedly, irrespective of their past consumption behavior (for example, new discs, movies, newspapers).

The first paper, to our knowledge, to consider the possibility of a collusive strategy based on alternating monopolies is Dougherty and Forsythe (1988), which looks more generally at the possibility of reproducing first best outcomes relaxing common knowledge assumptions. The authors consider a simplified problem in which there is no time discount and they find that alternating strategies can be a decentralized way to reach a collusive outcome sustainable without any side-payment. Omitting to put a time discount factor in the model seems restrictive, since any alternating strategy presumes some player is “waiting”. The discount factor will play an important role in our analysis, as in other intertemporal games.

The insight about the important of intertemporal market sharing has only been further developed very recently. There have been some empirical and experimental studies testing for the possibility of this kind of equilibria in a market situation. In particular, Einav (2002) develops an empirical analysis of strategies based on time release. The main motivation is that “as in many other situation of entry, product positioning and location choice, the release date decision is a trade-off between market size and market competition”. Einav (2002) analyzes the US motion picture industry and looks at the magnitude of switches in release date announcements, where a switch is defined as a change to the announced release date. Based on this information, he concludes that switches and films release dates are mainly strategic and not because of exogenous factors like production delays.

Zillante (2005) presents an experiment testing the emerging of time coordination in the baseball card market in US. Looking at the release dates of firms’ products, the author finds evidence of a tendency to space out products’ release dates.

From a theoretical point of view, Herings, Peeters, and Schinkel (2005) have contributed to the analysis of this issue by considering the emergence of an alternating monopoly as an equilibrium in a symmetric duopoly in which the Cournot equilibrium is also sustainable. They show that the alternating monopoly equilibrium can emerge, without employing any notion of collusive behavior, such as retaliation strategies. Firms just play a strategy of the form “entry at $t$ with probability 1 if you are not already in the market” and “exit at $t$ with probability 1 if you were in the market at $t - 1$”. This simple strategy naturally works in a symmetric duopoly. The equilibria has been numerically simulated for some values of the intertemporal discount factor $\delta$. The natural question is why a firm should play this game in a more complicated setting in which they can sustain more sophisticated decentralized collusive equilibria as in the classic tacit collusion models.
The aim of this paper is to illustrate particular situations in which firms’ preferred collusive strategies involves alternation. One of such situations is the Green and Porter (1988) framework, where demand is stochastic and firms cannot identify rivals’ deviation. Following this same intuition, Zillante (2005) compares one possible equilibrium with stochastic demand (N symmetric firms, retaliation period equal to \( T=1 \)) and he shows that there exists a range of discount factors for which alternating monopoly can be chosen over some standard collusive equilibria. This is not further developed, since the main aim of this paper is to perform the empirical investigation described above. We present here a more general analysis of alternating strategies which give a theoretical motivation of the emergence of AMS as opposed to alternative collusive strategies.

1 The model

In this section we formally define the supergame, the necessary assumptions and the implied information structure.

The relevant theoretical approach for analyzing firm’s interactions is the repeated game framework. To make the analysis simple, we build on the illustration of the Green and Porter (1984) model presented in Tirole (1988). We thus consider a market in which two symmetric firms compete in prices in a Bertrand framework.\(^7\) The goods are homogeneous and produced at a constant marginal cost \( c \), normalized to zero.

**Assumption 1** The game is repeated infinitely.

The periods are denoted by the index \( t \) and firms interact for an infinite number of periods.\(^8\)

**Assumption 2** Demand \( D \) is uncertain. In every period there are two possible states of nature, \( S \in \{H, L\} \) and demand realizations are i.i.d. over time.

\[
D \in \{D^H(p) = D(p), D^L(p) = 0\}, \quad \text{Prob}\{D = D^L(p)\} = \alpha
\]

Under Assumption 2, the underlying demand can be either in a “low state” or in a “high state”. With probability \( \alpha \) there is no demand (“low state”). With probability \( (1-\alpha) \), the demand is positive (“high state”) and equal to \( D(p) \). Assumption 2 aims describing in a simple way the features of markets in which the demand cannot be predicted by firms and random negative shocks occur.

We denote \( \pi(p) = pD(p) \) the profit of the firms in high state and we assume that it is strictly concave. We define the monopoly price \( p^m \), which formally corresponds

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\(^7\)We start considering only two firms. The N firms case is proposed in Section 3.

\(^8\)The assumption of infinite horizon can be generalized to the case in which the final date of the game is not known with certainty. A finite-horizon assumption, on the contrary, would imply the even stricter assumption that the terminal date is precisely and commonly foreseen.
to \( p^m \equiv \arg\max_p \pi(p) \). Consequently the monopoly profit is \( \pi^m = \pi(p^m) \). We also
suppose that there exists some \( \overline{p} \) such that \( D(\overline{p}) = 0 \) for \( S \in \{H, L\} \). We can restrict
the set of prices to the closed interval \( p \in [0, \overline{p}] \).

The following Assumptions 3 and 4 define the information structure of the game.

**Assumption 3** *Firms do not observe the realized demand state nor rivals’ prices or sales.*

Assumption 3 depicts a situation in which the firms do not have enough information to forecast demand and information on prices is either impossible to gather or too expensive. Firms infer prices on the basis of their own profit (or, equivalently, their ex post demands). When firms share the market equally and enjoy positive profit, they infer that they have both played the collusive price. However, when a firm observes zero profit, she doesn’t know the reason: a firm could be victim of “bad luck” (the state of nature is low) or of an unobservable price cut. The specification of a “low state” with zero demand is clearly an extreme simplification. However, this grants full tractability of the model and allows us to focus directly on the nontrivial signal extraction problem faced by the firms.

**Assumption 4** *At each period firms decide on entry. Entering involves a fixed cost \( f \). The entry decision (i.e. the payment of the fixed cost) is ex-post observable.*

This assumption captures the idea that firms have to necessarily undertake observable moves in order to enter the market and to reach consumers. We formalize this assumption stating that, at each period, firms have to pay a fixed cost \( f \) when entering the market. We assume that this action is observable at the end of the period (after demand realization)\(^9\).

For the clarity of the exposition, we consider the entry cost as negligible and set it equal to zero. We address the case of non-negligible entry costs in Section 3.3.

Under the previous assumptions, the repetitions of the game generate both a private and a public information history. For each firm, its private history is the sequence of its prices and sales across time. A public history is the sequence of information both firms observe. In this framework, at each period it is common knowledge whether the firms have entered the market (paying the entry cost). Moreover, the assumption that each firm observe its own profits implies that some other public information is also generated: all firms know if at least one firm makes zero profits (or zero ex post demand). In fact, if the “low state” of nature occurs, both

\(^9\)If firms could use the information about entry before price setting, they could deter any deviation under AMS. To make things interesting we assume that this is not possible. The firm active in the market cannot observe the presence of the rival before setting its price: the timing of each period is such that firstly a firm sets its price and only secondly it can observe the presence of the rival in the market. Therefore, we get the usual trade off between the short term benefits of deviation and the (optimal) punishment following detection.
firms make zero profit. If the “high state” of nature occurs but one firm undercuts, she knows that the competitor has to make zero profit. If finally one firm makes positive profits sticking to the collusive pricing strategy, it must be the case that the competitor has nonnegative profits (and then no firm has zero profits). As a consequence, the event “at least one firm has faced zero profit” is publicly observable in our model.

The public information described above allow firms to do some inference about the behavior of the competitors. In the following subsections, we analyze how traditional market sharing and alternating strategies respond to this information structure. We begin by studying the market sharing strategy. The analysis of alternating monopoly follows.

1.1 The market sharing strategy (MSS)

We first consider the benchmark case in which both firms are on the market at each period and play a market sharing strategy through price coordination. We know that, in an imperfect monitoring context, the optimal punishment strategy will not involve deterministic Nash reversion. The relevant collusive equilibrium has the following features: in the collusive phase, both firms enter at each period (i.e. pay the fixed cost) and coordinate on charging the collusive price until a firm makes zero profit. Once this public event occurs, Firms then coordinate on a randomization device and either jointly stay in the collusion phase or enter the punishment phase.\(^\text{10}\)

In order to characterize the optimal collusive strategy, we follow the approach of Abreu, Pearce, and Stacchetti (1986). The key insight of their contribution is that each stage of the infinitely repeated game can be represented as a static game in which the payoffs are decomposed in the sum of a stage game payoff and a continuation value (which represents the present value of future payoffs).

In the stage game, firm \(i\) (\(-i\), respectively) enters the market (paying the fixed cost) and charges a price \(p_i\) (or \(p_{-i}\)) in a finite and discrete action space \([0, \bar{p}]\).\(^\text{11}\) The profit function thus depends on the state of the demand \(S\) and on the prices played by the competitors. Consumers buy from the firm that sets the lower price and if the two firms charge the same prices, they share the market equally. The stage game has at least one Nash equilibrium, given by the static Bertrand equilibrium. Moreover, the minmax, \(\bar{\nu}\) of the repeated game is equal to zero. In the repeated game, firms maximize the sum of the expected discounted payoffs from the stage game. The discount factor \(\delta < 1\) is common knowledge.

We restrict the analysis to symmetric perfect public equilibria (SPPE) in pure strate-

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\(^\text{10}\)We use here the term punishment in an improper way, because in an imperfect monitoring framework low continuation values do not need to arise from a deviation. They are just a change of state aimed at providing the right incentives to the players.

\(^\text{11}\)The fact that the action space is discrete is a technical assumption which allows us to apply the framework developed in Abreu, Pearce, and Stacchetti (1986, 1990). Concretely, it implies that prices are discrete and not chosen in a continuum. Since in reality price has to be expressed in monetary units, this assumption is not considered as restrictive.
gies. A SPPE is a symmetric strategy profile in which, at any point in time, players do not rely on their private information but they base their decisions on public history. More formally, a public strategy for a firm is a mapping from the set of public histories to the set of actions. We take symmetric strategies and symmetry is used here in the strong sense that firms’ payoffs must be the same for any public history. Using the Abreu, Pearce, and Stacchetti (1986, 1990) approach, we transform the repeated game into an equivalent static game whose payoffs are the ones of the stage game augmented by a continuation payoff. We also allow for a public randomization device. In the stage game with public randomization, a realization of a public random variable is first drawn and observed by all players. Then each player chooses an action. This aims to represent a situation in which active collusion is formally forbidden and thus pursued through tacit coordination on a public signal. Under the hypothesis of public randomization, the continuation set is the convex hull of the set of values which can be supported by the pure strategies of the stage game. Then the set of payoffs supported by SPPE is compact, non-empty and convex. Formally, letting \( E_S(\delta, \alpha) \) be the set of payoff supported by SPPE, we have \( E_S(\delta, \alpha) = \{ v = 0, \tau(\delta, \alpha) \} \) and a SPPE that supports the ex-ante maximal payoff \( \tau(\delta, \alpha) \) always exists. We focus on this optimal SPPE as it is natural to assume that firms insist on this equilibrium whenever it is possible, because it insures highest payoffs.

The implementation of the optimal equilibrium \( \tau(\delta, \alpha) \) is obtained by using the so-called bang-bang property, which insures that it is possible to implement the optimal SPPE randomizing only between the two extremal points of the set \( E_S(\delta, \alpha) \). Firms start playing the stage game strategy \( p_m \) and, depending on the public information, with a certain probability they stick to that strategy or move to Nash reversion. Formally, the implementation of the optimal equilibrium is equivalent to solving the following problem:

\[
\tau(\delta, \alpha) = \max_{p, \beta, \gamma} v \\
\text{s.t. } v = (1 - \alpha)p_m^2 + \delta \{(1 - \alpha)[(1 - \gamma)\pi + \gamma \pi] + \alpha[(1 - \beta)\pi + \beta \pi]\} \\
v \geq (1 - \alpha)p_m^2 + \delta[(1 - \beta)\pi + \beta \pi] \\
\beta, \gamma \in [0, 1]
\]

where \( \gamma \) and \( \beta \) are the conditional probabilities of entering into a perpetual Nash

\(^{12}\)Our framework differs slightly from the one in Abreu, Pearce, and Stacchetti (1986, 1990). In fact they rely on the hypothesis of full support of the distribution of the shocks, which means that every public event can arise with positive probability for any profile of actions. In our case, the event “firms share the market in halves” never arises for any asymmetric action profile \( (p_i \neq p_{-i}) \). This does not affect the main point: at every symmetric profile every history can arise. It is not possible to perfectly detect deviations. To reconnect our analysis to the approach of Abreu, Pearce, and Stacchetti (1986, 1990), it is sufficient to assume that, with a positive probability \( \rho \) consumer do not detect the price asymmetry and firms still share the market in halves. If this probability is close to zero, all our results apply.
reversion given the state of nature (respectively, H or L). \( \pi^m \) is the monopoly profit obtained when firms optimally insist on price \( p^* = p^m \). Condition 2 is an incentive compatibility constraint. Condition 1 is the stage game payoff when all players play \( p^m \) augmented by the expected continuation payoff defined by the bang-bang implementation.

The maximal payoff \( \pi(\delta, \alpha) \) is obtained with the following equilibrium strategies:

**Proposition 1** Firms optimally set \( \gamma^* = 0 \) and \( \beta^* = \frac{1-\delta}{\delta(1-2\alpha)} \). If the realization of the profit is \( \pi^m/2 \) and firms are in the collusive phase, firms don’t punish. When the realization of the profit is 0, firms switch to Nash reversion with probability \( \beta^* \).

**Proof.** See appendix. ■

Proposition 1 describes firms’ optimal strategy. Firms never find it optimal to punish when they observe high profits. On the contrary, when they observe 0 profits, they optimally punish with a certain probability \( \beta^* \), to give the right incentives to sustain the collusive equilibrium. If \( \alpha > \frac{1}{2} \) the market sharing equilibrium does not exist (i.e. for all \( \delta \leq 1 \) there does not exist a probability \( \beta \) capable of enforcing collusion).

**Proposition 2** The incentive constraint in the collusion phase always binds. The MSS is sustainable if \( \delta \geq \delta(\alpha) = \frac{1}{2(1-\alpha)} \) and the firms’ optimal net present value is \( \pi(\delta, \alpha) = \frac{\pi^m(1-2\alpha)}{1-\delta} \).

**Proof.** See appendix. ■

It is worth noticing that this equilibrium never exists when demand uncertainty is high (\( \alpha > 1-\frac{1}{2\delta} \)). For \( \alpha \leq 1-\frac{1}{2\delta} \), tacit collusion is sustainable with coordination on a market sharing rule.

### 1.2 The Alternating Monopoly Strategy (AMS)

We now consider, in the same framework as before, an equilibrium in which firms in the collusive phase alternate symmetrically their presence in the market.\(^\text{13}\)

This strategy allows for a different use of the available information. Ex-post observability of entry (Assumption 4) allows each firm to solve the signal-extraction problem. We assume here perfect public observability of the entry move. This is a simplified way to describe a situation in which monitoring entry is easier than monitoring prices. In their periods of activity, firms optimally set the price at \( p^m \). The net present value of firm 1 and 2 are respectively:

\(^\text{13}\)The optimality of the particular arrangement “one period in-one period out” is not discussed here. Given the symmetric nature of the two firms, it seems reasonable to restrict attention to this particular rule.
\[ v_1 = (1 - \alpha)\pi^m + \delta^2 v_1 \]  
\[ v_2 = (1 - \alpha)\delta\pi^m + \delta^2 v_2. \]  

Due to the cost of the time lag before the second firm is active, the two firms face different incentive constraints. Obviously there is no incentive to deviate for the firm which is expected to be alone in the market in the first period under the collusive agreement (firm 1). The incentive constraint which matters is the one of firm 2.

\[ v_2 \geq (1 - \alpha)\pi^m. \]  

In an alternating monopoly, a deviation consists in a firm’s decision to be unexpectedly active in the market during the period not allocated to her under the tacit agreement. In case of deviation, each firm chooses its best profitable deviation, \( \tilde{\mu}_i = p^m_i - \epsilon \), stealing all the demand (when positive) from the competitor and making monopoly profit. In this case, the deviating firm enjoys the expected monopoly profit yielded by undercutting its competitor and in the subsequent periods it suffers from retaliation that lasts infinitely. Under AMS, a deviation from collusion is detected with probability 1, because the presence in the market at the “wrong” time is observable. Then maximal punishment (at the minmax continuation value) is optimal and it enforces collusion.

The equilibrium is characterized by the following Proposition.

**Proposition 3** AMS is sustainable if \( \delta \geq \delta_{AMS} = \frac{\sqrt{5} - 1}{2} \). The two firms’ optimal net present values are \( \pi_1(\delta, \alpha) = \frac{(1 - \alpha)\pi^m}{1 - \delta} \) and \( \pi_2(\delta, \alpha) = \frac{(1 - \alpha)\pi^m\delta}{1 - \delta} \).

## 2 Comparison of the collusive strategies

In order to discuss the characteristics of the two collusive equilibria, we start by considering the two feasibility thresholds \( \delta_{AMS} \) and \( \delta \). We notice that \( \delta_{AMS} \) doesn’t depend on the demand uncertainty parameter \( \alpha \). As explained above, the activity of firms is public information and therefore alternation solves the firms’ signal-extraction problem. However, the equilibrium requires firms to wait in non-releasing periods. This has a cost which depends on the level of time impatience, measured by the discount factor \( \delta \). There is thus a trade-off between the cost of the inefficiency related to signal-extraction and the cost of being out of the market for one period out of two.

As previously said, the market sharing equilibrium exists for level of \( \alpha \) strictly smaller than 1/2. The existence of AMS equilibrium does not depend on the value of \( \alpha \).

There exists a full range of \( \alpha \) for which both equilibria exist, depending on the degree of impatience of the firms. When there is no uncertainty of demand (\( \alpha = 0 \), the
market sharing equilibrium is more effective and easier to sustain ($\hat{\delta}(0) < \delta_{AMS}$). The burden of alternation is not compensated by any other gain. On the other hand, when demand uncertainty is very high, the burden of waiting one period is offset by the gain related to the resolution of the signal-extraction problem.

The results are summarized in the following Propositions.

**Proposition 4** For $\alpha > \frac{1}{2}$ the MSS is never feasible. AMS is feasible iff $\frac{\delta}{1-\delta^2} \geq 1$.

- If $\frac{1}{2(1-\alpha)} \leq \delta \leq \frac{\sqrt{5}-1}{2}$ only MSS is feasible.
- If $\frac{\sqrt{5}-1}{2} \leq \delta \leq \frac{1}{2(1-\alpha)}$ only AMS is feasible.
- If $\delta \geq \max \{\frac{\sqrt{5}-1}{2}, \frac{1}{2(1-\alpha)}\}$ both equilibria are feasible.

In order to select one equilibrium in the region in which both equilibria are feasible, we adopt a Pareto criterion. In the alternating monopoly, the net present values of firms are different: we have $v_2 \leq v_1$. Then AMS Pareto dominates MSS if and only if $v_2$ is bigger than $v_1$.

**Proposition 5** For $\delta \geq \max \{\frac{\sqrt{5}-1}{2}, \frac{1}{2(1-\alpha)}\}$, AMS is preferred to market sharing whenever $2\alpha + \delta > 1$.

The results given in the Propositions 4 and 5 are illustrated in Figure 1.

![Figure 1: Regions of Equilibria](image-url)

In Region I only MSS is feasible. For small values of $\alpha$, the signal extraction problem ("retaliation or bad luck") of the benchmark agreement is not severe. The discount $\delta$ is not big enough to make an alternate equilibrium feasible.

In Region IV the opposite holds. The demand shock, and consequently the signal

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extraction problem, is severe and MSS is not sustainable. Only AMS is sustainable. Firms are willing to alternate solving the information problem which causes the failure of the MSS. If \(2\alpha + \delta > 1\), the net present values of AMS is bigger than the one of MSS. This condition splits in two the hybrid region, delimiting Region II and Region III (dashed line). Region II is the area in which MSS dominates AMS: \(\alpha\) is relatively small and the probability of inefficient retaliation is small. In Region III the opposite holds. The result relies on the trade-off between uncertainty (measured by \(\alpha\)) and impatience (measured by \(\delta\)). In the case of market sharing, retaliation occurs in equilibrium: the cost of inefficient retaliation depends on the magnitude of \(\alpha\). In the case of AMS, firms have to wait in some periods, which is costly if the discount rate is smaller than one. When \(\delta \to 1\) and time impatience is not an issue AMS always do better than MSS.

### 3 Robustness checks

The results obtained so far point out the existence and efficacy of AMS under some assumptions on the technology, the market conditions and the information structure. The aim of this section is to test the robustness of the model to some natural extensions.

#### 3.1 N Firms

One natural extension of the basic model is to consider a larger number of firms in the industry and in the collusive agreement. In general, an increase in the number of participants weakens collusive agreements. Due to its peculiarities, the impact of a less concentrated market could have a harsher detrimental effect in the AMS equilibrium than in the MSS. Under AMS, the participants must stay out of the market for a number of periods which is increasing in the number of colluding firms: for this reason the restriction to two players could be particularly favorable to AMS. This extension investigates this issue.

We suppose the market is less concentrated, the number of firms is now equal to \(N\). Intuitively, the collusive equilibrium is more difficult to sustain because the industry monopoly profit has to be shared among a larger number of players, making deviation more profitable. Therefore, the incentive to deviate is stronger the greater the number of firms in the market. The threshold above which collusion is sustainable under MSS model becomes:

\[
\delta(N, \alpha) = \frac{N - 1}{N(1 - \alpha)}
\]  

Equation (6) is a generalization of the stylized model when only two firms are in the market. The net present value is thus:
The AMS equilibrium is affected in the same way. In analogy with the duopoly case, we take simple symmetric alternating strategies in which each firm enters in turn for one period and then waits until each firm has been in the market for one period. If there is entry of two firms at the same period, retaliation occurs. With \( N \) players, firms have to wait longer and it is more difficult to satisfy the incentive compatibility constraint. The threshold above which collusion is sustainable in an AMS equilibrium is determined implicitly by the following equation:\(^{14}\)

\[
\delta_{AMS}^{N-1} = 1 - \delta_{AMS}^N.
\]

The net present value of the firm who stays on the market in the \( n^{th} \) period is equal to:

\[
\bar{v}_n(\delta, \alpha, N) = \frac{(1 - \alpha)\pi^m \delta^{n-1}}{1 - \delta^N} \quad n = 1, ..., N
\]

Once again, we compare the feasibility conditions and the net present value of the firms under the two collusive strategies.

**Proposition 6** The thresholds \( \delta_{AMS}(N) \) and \( \delta(N) \) are both increasing functions of the number of firms \( N \). However, the impact of an increase in the number of firm on the feasibility of collusion is more severe in the case of MSS than in the case of AMS. We have:

\[
\frac{\partial \delta(N, \alpha)}{\partial N} > \frac{\partial \delta_{AMS}(N)}{\partial N}.
\]

Moreover, increasing the number of firms decreases the threshold level \( \alpha(\delta) \) above which AMS is preferred to MSS.

**Proof.** See appendix. ■

In line with the existing literature we find that an increase in the number of players make collusion more difficult. However, AMS is more robust to an increase in the number of participants. This is the case for the “naive” alternating strategy we have presented in this extension (each player is active in the market as a monopolist one period out of \( N \)). This is probably not the optimal alternating strategy with \( N \) firms. However, choosing a better alternating strategy would only reinforce the result that AMS suffers less from the introduction of more players.

\(^{14}\)The equation comes from the incentive constraint of the last firm active in the market. This is the most stringent condition to meet, therefore it is the relevant one.
3.2 Observable stochastic demand shocks

Another extension\textsuperscript{15} of the basic model may allow for observable stochastic shocks. This exercise is reminiscent of the one of Rotemberg and Saloner (1986)\textsuperscript{16} and it is a first attempt to embed stochastic demand shifts in a framework with unobservable demands and prices. We now assume that \( \alpha \) is a random variable with a known distribution \( F(\alpha) \) on the support \([\underline{\alpha}, \overline{\alpha}]\). At each period, the magnitude of the shock \( \alpha \) on the current period is observable. Future shocks are independent and identically distributed across periods and the distribution \( F(\alpha) \) is common knowledge. Let \( \mu_\alpha \) be the expectation of \( \alpha \), i.e. \( \mu_\alpha = \int_\underline{\alpha}^{\overline{\alpha}} \alpha \, dF(\alpha) \).

We first consider the MSS. In this case, the value of sustaining collusion can be written:

\[
\tau(\delta, \alpha) = \frac{(1 - \alpha)\pi^m}{2} + (1 - \alpha)\delta \hat{v}(\delta, \mu_\alpha) + \alpha(1 - \beta)\delta \hat{v}(\delta, \mu_\alpha)
\]

where \( \hat{v}(\delta, \mu_\alpha) \) is the (expected) continuation value of collusion.

\[
\hat{v}(\delta, \mu_\alpha) = \frac{(1 - \mu_\alpha)\pi^m}{1 - \delta[(1 - \mu_\alpha) + \mu_\alpha(1 - \beta)]}
\]

Moreover, the incentive compatibility constraint is:

\[
\tau(\delta, \alpha) \geq (1 - \alpha)\pi^m + \delta(1 - \beta)\hat{v}(\delta, \mu_\alpha)
\]

Substituting for the value of \( \tau(\delta, \alpha) \) as defined in (8) this simplifies to:

\[
\delta \hat{v}(\delta, \mu_\alpha) \beta \geq \frac{\pi^m}{2}
\]

As one can see from Equation (9), the incentive compatibility constraint does not depend on the realized value of \( \alpha \). Solving the problem as in Section 1.1, we obtain a feasibility threshold of the form:

\[
\delta(\mu_\alpha) = \frac{1}{2(1 - \mu_\alpha)}
\]

We now turn to the case of AMS. In this case, the value \( v_2 \) just depends on \( \mu_\alpha \) while the short run gain from deviation depends on the observed realization of \( \alpha \). The incentive compatibility constraint becomes:

\[
v_2 \geq (1 - \underline{\alpha})\pi^m
\]

where \( v_2 = \delta \int_\underline{\alpha}^{\overline{\alpha}} (1 - \alpha)\pi^m dF(\alpha) + \delta^2 v_2 \) and \( \underline{\alpha} \) is the lower bound of \( \alpha \). Since at each time \( t \) the level of \( \alpha \) is observable, the incentive compatibility constraint has to be satisfied for all possible realizations of \( \alpha \), even when the short run gains from

\textsuperscript{15}We thank Joseph Harrington for suggesting this extension of the model.

\textsuperscript{16}Our model only differs from theirs because it is always possible to sustain the monopoly price for every \( \alpha \).
deviation are maximal (low values of $\alpha$). Intuitively the bigger the dispersion of $\alpha$
(and thus the smaller $\alpha$ with respect to $\mu_\alpha$), the more difficult is to sustain AMS
collusive equilibrium. Formally, the feasibility threshold has the form:

$$\delta_{AMS}(\alpha, \mu_\alpha) = \frac{\sqrt{(1 - \mu_\alpha)^2 + 4(1 - \alpha)^2} - (1 - \mu_\alpha)}{2(1 - \alpha)}$$

Contrary to the basic case of deterministic $\alpha$, in this case the AMS feasibility
threshold is not constant but increasing in $\mu_\alpha$. The feasibility of MSS, however, has
the same shape as before (except that it is now a function of $\mu_\alpha$). Therefore, in the
case of observable stochastic shocks, AMS becomes relatively less attractive with
respect to the basic case. Under MSS at each period the firms’ choice of deviating
or sticking to the collusive is independent of $\alpha$. On the contrary, in the AMS, firms
might want to deviate when the present gain is high compared to the expected net
present value of collusion.

### 3.3 Non negligible fixed cost

We now look at the case of non negligible fixed costs. In this case, a firm has to
pay the cost $f$ at each period if she wants to enter the market. In the MSS case,
firms pay $f$ at each period. In the case of AMS, they just pay $f$ one period out of
two. For this reason, the presence of a fixed cost of entry affects differently the two
equilibria.

The feasibility threshold in the MSS case is computed as in Section 1.1. This gives:

$$\delta \geq \delta_{MSS}(\pi^m, \alpha, f) = \frac{1}{2(1 - \alpha) - f/\pi^m}$$

As it appears from Equation 10, the presence of a non negligible entry cost
increases the relevant threshold, making MSS more difficult to sustain. The reason
is that the incentive to deviate from the collusive pricing strategy is higher when
the fixed cost is large.

On the contrary, the relevant feasibility threshold in the case of AMS is not affected
by an increase in $f$ and is still given by Equation 3, which gives a threshold level:

$$\delta \geq \delta_{AMS} = \frac{\sqrt{5} - 1}{2}$$

As for the values of the firms, in the two cases the presence of a non negligible
fixed cost of entry reduces the stream of profits of the firms. In the case of MSS we
have:

$$\pi(\delta, \alpha) = \frac{\pi^m(1 - 2\alpha)}{1 - \delta} - \frac{1}{1 - \delta}f$$

In the case of AMS the values of the two firms are respectively:
As intuitive, the loss is smaller in the case of MSS, since the fixed cost is paid only half of the periods. This means that the AMS collusive rule improves the total industry profits with respect to the standard collusive setting. Curiously, avoiding the duplication of fixed costs, AMS also increases total welfare. However, this consideration introduces issues that are out of the scope of the present paper and that should be considered in a broader framework.

Concluding, with respect to the situation illustrated in Figure 2, the presence of non negligible fixed costs shifts downwards the two curves delimitating the feasibility of MSS and the Pareto ranking of the two equilibria. It makes AMS both easier to sustain and more desirable for firms.

4 Discussion and conclusions

Our analysis origins from the observation that in many markets firms’ strategies display forms of intertemporal alternation. In the literature, this kind of behavior has been mainly explained as the solution of a differentiation problem, as illustrated in Bester, de Palma, Leininger, Thomas, and von Thadden (1996). In their framework, temporal coordination is simply interpreted as a differentiation strategy aimed to solve a coordination problem. Our model offers a different and complementary explanation for this behavior, stressing the consequences of alternation on the competitive outcome. We show that alternation may relax competition and can be interpreted as a tacit collusion device. In our analysis, the choice of alternating strategies depends on some peculiarities of the competitive environment, which make alternation preferred to other possible coordination strategies. Contrarily to Herings, Peeters, and Schinkel (2005), we consider a market in which, in addition to the existence of a non-cooperative equilibrium, other collusive outcome are also sustainable.

An important point is that the choice of alternating strategies is not driven here by technological reasons. Intuitively, one could expect the difference between the collusion schemes to depend on the shape of the cost function, and in particular whether the payoff Pareto frontier is convex or concave (with sharing and alternating preferred in each case). However, our result does not depend on the particular form of the cost function, but on the characteristics of the environment in which firms compete. In particular, we stress how the presence of an observable move (entry), distinct from the competitive stage (price setting), can serve as a coordination device, reducing the cost of monitoring in incomplete information environments. In this sense, the model show how apparently inefficient strategies, such as alternation when market sharing is also feasible, can indeed be very effective in certain
environments. When firms collude in the presence of imperfect monitoring, the collusive agreement may prescribe actions that tend to minimize the cost of monitoring. This main theoretical insight may go well beyond the specific problem illustrated in the paper: in our model, this behavior takes the alternating monopoly structure. In further research, this insight could be put forward and applied to more general frameworks. For instance, the proposed model of alternating monopoly could be reinterpreted as to explain rather different phenomena, such as the strategic behavior of firms operating in industries in which competition is driven by innovation and quality improvements. This could be formalized following the approach proposed by Fudenberg and Tirole (2000). In their model, at each period $t$ a firm can innovate by introducing a better and incompatible technology that has an (exogenous) value for consumers given by $\eta(t+1)$. Innovation is permanent and everyone’s technology is improved by $\eta$ from period $t+1$ on. However, due to the presence of a competitive fringe, profits from innovations last only one period and they are driven to zero afterwards. In this setting, firms that innovate enjoy the exogenous value of innovation: if two or more firms innovate at the same time, they have to share the value of the innovation (the total willingness to pay for innovation is fixed at $\eta$). If we let $(1-\alpha)$ be the probability of adoption of the innovation by consumers, this drives us back to our original framework and the all the results of our analysis apply.

On a more practical ground, our paper helps to identify situations in which the difficulty of observing prices may not be as an obstacle to collusion as previously thought: colluding firms can deploy alternative collusion schemes, some of which have not received attention in the literature so far. We concentrate on the possibility of colluding using alternating strategies. We show that this method of collusion is made easier if the presence of competitors in the market (entry) is easily observable, due for instance to marketing strategies providing easy or very cheap information, as in the case of advertising campaigns. Taking into account this kind of strategies can have important policy implications. In particular, the potential for tacit coordination is usually taken into account in merger policy. The results of this paper suggest to policymakers that evidence of alternating strategies may indicate the possibility of tacit collusion and thus be considered for the analysis of the coordinated effects of mergers. As mentioned above, alternating strategies can have different explanations: they can solve coordination problems or, in a collusive framework, be a response to inefficiencies in the monitoring technology. The empirical distinction of these two scenarios is not necessarily straightforward. Our analysis contributes to the understanding of this issue stressing the characteristics of markets which are more vulnerable to the anticompetitive effects of alternating strategies. These are concentrated markets in which price monitoring is costly but firms may coordinate relying on other observable actions. If products are homogeneous, this seems to be a very reasonable way to relax competition, which deserves some attention when evaluating the competitive potential of market.

In conclusion, the application of the alternating monopoly model to different situ-
ations seems to be promising in contributing to the explanation of complex firms strategies in environments characterized by uncertainty and imperfect information.

References


Appendix

Proof of Proposition 1.

From Equation (1) we can express the value of $v$ as:

$$v = \frac{(1-\alpha)\pi_m}{2 - \delta[(1 - \alpha)(1 - \gamma) + \alpha(1 - \beta)]}$$

The definition of $v$ lead to the following expression of the derivatives with respect to $\gamma$:

$$\frac{\partial v}{\partial \gamma} = -\frac{\delta \pi_m (1 - \alpha)}{(1 - \delta[(1 - \alpha)(1 - \gamma) + \alpha(1 - \beta)])^2} < 0$$

Then $v$ is decreasing in $\gamma$.

From Equation (2), we rewrite the incentive constraint as:

$$f(\gamma, \beta, \delta, \alpha) = (1 - \alpha)\pi_m + v\delta(1 - \beta) - 1) + \delta v \beta \leq 0$$

By taking derivative of the incentive constraint $f(\gamma, \beta, \delta, \alpha)$ with respect to $\gamma$, we obtain:

$$\frac{\partial f(\gamma, \beta, \delta, \alpha)}{\partial \gamma} = \frac{\partial v}{\partial \gamma}(\delta(1 - \beta) - 1) > 0$$

Now define the Lagrangian:

$$L = v(\gamma, \beta, \delta, \alpha) - \mu f(\gamma, \beta, \delta, \alpha) + \lambda \gamma$$

where $\mu$ is the multiplier associated to the incentive constraint $f(\gamma, \beta, \delta, \alpha)$ and $\lambda$ is the multiplier associated with the constraint $\gamma \geq 0$.

By taking derivatives of the Lagrangian with respect $\gamma$ we obtain:

$$\lambda = -\frac{\partial v}{\partial \gamma} + \mu \frac{\partial f}{\partial \gamma} > 0$$

This implies that $\gamma^* = 0$.

Now, using (1) and Equation (2) we obtain the condition:

$$\delta(v \! - \! v)(\beta - \gamma) \geq (1 - \alpha)[\pi_m - \frac{\pi_m}{2}]$$

Given that $v \! - \! v > 0$ and $\gamma^* = 0$, we have $\beta > \gamma^* = 0$, which implies $\beta^* > 0$. ■

Proof of Proposition 2.

From Equation (1) we can obtain the value of $v$ which gives the following derivative with respect to $\beta$:
\[ \frac{\partial v}{\partial \beta} = -\frac{\alpha \delta \pi^m (1 - \alpha)}{(1 - \delta [(1 - \alpha)(1 - \gamma) + \alpha(1 - \beta)])^2} < 0 \]

From Equation (2), we define:
\[ f(\beta, \delta, \alpha) = (1 - \alpha)\pi^m + v(\delta(1 - \beta) - 1) + \delta v \beta \]

By taking derivative of the incentive constraint \( f(\beta, \delta, \alpha) \) with respect to \( \beta \), we obtain:
\[ \frac{\partial f(\beta, \delta, \alpha)}{\partial \beta} = -\delta \beta v + \frac{\partial v}{\partial \beta}(\delta(1 - \beta) - 1) \]
\[ = -\frac{(1 - \alpha)^2 \delta(1 - \delta(1 - \gamma)) \pi^m}{(1 - \delta [(1 - \alpha)(1 - \gamma) + \alpha(1 - \beta)])^2} < 0 \]

By taking derivatives of the Lagrangian with respect \( \beta \) and observing that the \( \frac{\partial v}{\partial \beta} < 0 \) and \( \frac{\partial f(\beta, \delta, \alpha)}{\partial \beta} < 0 \), we obtain:
\[ \mu = \frac{\partial v}{\partial \beta} > 0 \]

We can conclude that the incentive constraint is binding providing that \( \beta^* \leq 1 \) when \( \gamma^* = 0 \). The condition is thus:
\[ \beta^* = \frac{1 - \delta}{\delta(1 - 2\alpha)} \leq 1 \]

which implies the condition on \( \delta \):
\[ \delta \geq \frac{1}{2(1 - \alpha)} \]

\[ \square \]

**Proof of Proposition 5.**

We compare the value of the firm in the MSS case, \( v \), with the one of AMS, \( v_2(y(p^m_m, \alpha)) \). In the AMS case we focus on the second firm playing since it is the more stringent case to analyze. We compute them for the two periods to have consistency in the values.

The value of the firm in the benchmark case for two periods is thus:
\[ v = \frac{(1 - \alpha)\pi^m}{2} + \frac{\pi^m}{2}(1 - \alpha)\delta ((1 - \alpha)(1 - \gamma) + \alpha(1 - \beta)) + \]
\[ \delta^2 v ((1 - \alpha)(1 - \gamma) + \alpha(1 - \beta))^2 + \]
\[ \delta v ((1 - \alpha)\gamma + \alpha \beta) + \]
\[ \delta^2 v ((1 - \alpha)(1 - \gamma) + \alpha(1 - \beta)) ((1 - \alpha)\gamma + \alpha \beta) \]
and in the AMS

\[ v_2 = (1 - \alpha)\delta \pi^m + \delta^2 v_2 \]

Define \( A = ((1 - \alpha)(1 - \gamma) + \alpha(1 - \beta)) \) and \( B = ((1 - \alpha)\gamma + \alpha\beta) \). Note that when \( \gamma = 0 \) and \( \beta = 0 \), \( A = 1 \) which implies \( B = 0 \).

By subtracting the two definitions of \( v \) and \( v_2(y(p_2^m, \varnothing)) \) we obtain:

\[
\begin{aligned}
  v - v_2 &= (1 - \alpha)\pi^m \left( \delta \frac{1}{2} A - \frac{1}{2} \right) + \delta^2 v_2 A^2 \\
  &\quad - \delta^2 v_2 + \delta v B + \delta^2 v AB
\end{aligned}
\]

Given that \( \alpha > 0 \) and \( v > v_2 \), for \( \delta \to 1 \) it is possible to see that the only way to obtain \( v = v_2 \) is by setting \( A = 1 \) (\( \gamma = 0 \) and \( \beta = 0 \)). But this condition is not incentive compatible since we know that in equilibrium it is optimal for the firm to maintain the probability to punish strictly positive, which means \( \beta^* > 0 \). Therefore, for \( \beta^* > 0 \), \( A < 1 \) and \( B > 0 \) which implies that, when \( \delta \to 1 \), \( v < v_2 \).

**Proof of Proposition 6.**

Consider first the threshold of the MSS case: \( \hat{\delta}(N, \alpha) = \frac{N - 1}{N(1 - \alpha)} \). We have:

\[
\frac{\partial \hat{\delta}(N, \alpha)}{\partial N} = \frac{1}{(1 - \alpha)N^2} > 0
\]

In the case of alternating monopoly \( \hat{\delta}(N) \) is implicitly defined by Equation (7). Applying the implicit function theory we derive:

\[
\frac{\partial \hat{\delta}_{AMS}(N)}{\partial N} = -\frac{\delta(1 + \delta) \log(\delta)}{N(1 + \delta) - 1} > 0
\]

We also have to show that \( \frac{\partial \delta(N, \alpha)}{\partial N} > \frac{\partial \hat{\delta}_{AMS}(N)}{\partial N} \). This is equivalent to showing:

\[
\frac{\partial \hat{\delta}_{AMS}(N)}{\partial N} - \frac{\partial \delta(N, \alpha)}{\partial N} \equiv \Delta(N) < 0
\]

Indeed, we have:

\[
\Delta(N) \bigg|_{N=2} = \frac{\delta(1 + \delta) \log(\delta)}{2(1 + \delta) - 1} + \frac{1}{4(1 - \alpha)} < 0
\]

\[
\frac{\partial \Delta(N)}{\partial N} = \frac{2}{N^3(1 - \alpha)} - \frac{\delta(1 + \delta) \log(\delta)}{(N(1 + \delta) - 1)^2} > 0, \forall N \geq 2
\]

\[
\lim_{N \to \infty} \Delta(N) = 0
\]

Finally, we have to show that the threshold value \( \alpha(\delta) \) above which AMS is preferred to the benchmark is decreasing in \( N \). We have:

\[
\alpha(\delta) = \frac{\delta - N\delta^N + (N - 1)\delta^{N+1}}{N(\delta - \delta^N)}
\]

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Then:

$$\frac{\partial \alpha(\delta)}{\partial N} = \frac{\delta(1-\delta^N)^2 + N^2(1-\alpha)(1-\delta)\delta^N \log(\delta)}{N^2(\delta - \delta^N)^2} \equiv \Gamma(N)$$

To determine the sign of $\Gamma(N)$, it is sufficient to notice that:

$$\Gamma(N)\bigg|_{N=2} = -\frac{1-\delta^2 + 2\delta \log(\delta)}{4(1-\delta)} < 0, \quad \forall \delta \in [0, 1]$$

$$\frac{\partial \Gamma(N)}{\partial N} = \frac{\delta(2(\delta - \delta^N)^2)(1-\delta^N) - N(1-\delta)\log(\delta) - (N-1)N(\delta + \delta^N)\log(\delta)}{N^3(\delta - \delta^N)^3} > 0, \quad \forall N \geq 2$$

$$\lim_{N \to \infty} \Gamma(N) = 0$$