Optimal Dynamic Management of a Renewable Energy Source under Uncertainty

Catherine Bobtcheff
Toulouse School of Economics (CNRS and LERNA)

Abstract

I consider a risk averse and prudent social planner who has access to different energy sources to produce electricity: hydroelectricity produced with a dam and thermal electricity with unlimited supply at some exogenous cost. The dam is supplied with a random water flow. The presence of constraints on a minimal and on a maximal storage capacity makes electricity consumption smoothing possible only when the quantity of water available to the agent lies in a certain range that I determine. Consumption smoothing is possible even when the dam is almost empty thanks to the alternative costly energy source. Moreover a comparative static analysis reveals that the marginal propensity to produce hydroelectricity is an increasing function of the cost of the second technology. Therefore, the availability at a low cost of the fossil source improves time diversification. Finally, the optimal electric park is composed of a number of dams that is increasing with the cost of the second technology.

Keywords: intertemporal expected utility maximization, hydroelectricity, thermal power, investment under uncertainty, prudence.

JEL Classification: C61, Q25, Q42.

*Acknowledgements: I am particularly grateful to my supervisor Professor Christian Gollier for his help and his advice. I also would like to thank Alain Ayong Le Kama, Gilles Lafforgue, and Katheline Schubert for helpful comments. E-mail: catherine.bobtcheff@univ-tlse1.fr.
1 Introduction

The aim of this work is to determine the optimal structure of an electric park that generates power with different energy sources. To preserve the environment and energy sources that are exhaustible, governments are increasingly concerned with the use of renewable energy sources besides classic thermal power sources. However, renewable energy sources are not easy to use as their availability is not constant over time. Therefore, an electric park must be designed by taking into account this random availability and its management has to solve the problem of providing enough electricity even when renewable sources are not available in the short run.

Norway, for instance, is the sixth largest hydropower generator in the world and the biggest in Europe. Hydropower accounts for 99% of the electricity generated and annual production varies to a great extent in line with precipitation levels. Thus when the country faces dry periods as it was the case in 2002 and 2003, hydropower reservoirs work as buffers between output and consumption. Besides hydropower, electricity is also generated from sources such as natural gas and wind. Indeed, “gas-fired power station” can be started up and closed down at short notice. They are suitable for providing peak-load power but have a relatively high cost. In fact, during dry periods, the loss of hydropower output is offset by increasing thermal power generation.\footnote{See Ministry of Petroleum and Energy of Norway (2004).}

In this work, I focus on two energy sources. The renewable energy source has a random availability whereas the thermal power source is available at an exogenous market price. The possibility to store the renewable energy source in a dam allows to smooth consumption over time. During a dry episode, some of the water stored in the dam is consumed and the water reserve goes down, potentially to the lower limit of the reservoir. In that case, electricity consumption may be limited or rationed. On the contrary, when the water inflow is higher, the dam is replenished potentially up to the maximum capacity of the reservoir. Therefore, the dam’s capacity is a key factor of the optimal management policy as the Norwegian example illustrates. Besides this renewable energy source, the permanent availability of thermal power softens the effect of uncertainty of the water inflow.

A large body of literature concerning commodity storage presents meaningful results. Williams and Wright (1982, 1984) developed a model where supply is stochastic and where production and storage are performed by competitive profit maximizers. They found that “storage is much more effective in eliminating excessive levels of consumption and low prices than in preventing low levels of production and high prices”. They explained this result by evoking the non symmetry of storage. Indeed, storage has to be non negative meaning in this case that water cannot be “borrowed” during a drought. Deaton and Laroque (1992, 1996) also worked on the topic of commodity prices and commodity storage. In the second paper, they tried to explain some stylized facts of commodity price behavior by fitting a competitive storage model directly to the data. They proposed two ways to model productivity shocks: either iid shocks or time dependent (autoregressive) shocks. But finally none of the two models fits the data well. Deaton and Laroque explained this failure as follows: “all the autocorrelation in the data has to be attributed to the underlying processes. Although speculation is capable of increasing the autocorrelation that would otherwise exist in an unmoderated price series, it cannot raise it to the levels that we observe”.

The literature dealing with the use of energy sources has expanded in many directions.
Garcia et al. (2001) analyzed the price formation process and its policy implications in an infinite horizon duopoly model. They focused on two hydroelectricity producers who engage in dynamic Bertrand competition. At each date, water reservoirs are replenished with some strictly positive probability and a price cap affects the opportunity cost of producing electric power. They found that hydroelectricity producers might sell less today to have more capacity tomorrow: they adopt a strategic pricing behavior. They explained that the introduction of a price cap may shift down the entire price distribution. Crampes and Moreaux (2001) studied a model where two energy sources are available: hydroelectricity and thermal electricity. They focused on a model of two time periods and did not introduce uncertainty in the hydroelectric technology. They studied the case of a central planner, of a monopoly that is regulated or not and the case of Cournot competition. They concluded that in the presence of hydroelectricity, thermal plants have to be dynamically planned. Moreover the optimized output for the thermal station is determined by the intertemporal specification of utility and costs. Hotelling (1931) studied the dynamic pricing of a non-renewable and exhaustible natural resource and he found that, in the competitive case without stock externality, the price of this resource must grow at a rate which is equal to the interest rate of the financial market. Extended to the problem of a social planner who have to determine the optimal use of the resource, this rule implies that the optimal consumption declines over time at a rate corresponding to the ratio of the social discount rate and the elasticity of marginal utility of consumption. Optimality also requires that the resource must be exhausted asymptotically. As Heal (1993) noted in his review on the optimal use of exhaustible resource, many extensions to this initial model have been explored. Hoel (1978), for instance, introduced uncertainty in a setting with two energy sources: the date when the substitute will become available is known, but its unit cost is uncertain. He found that an increase in uncertainty may increase the consumption depending on the shape of the utility function. Ayong Le Kama (2004) studied the use of a unique energy source under uncertainty in a finite horizon model. He found that introducing two types of constraints, one on the availability of the resource and another on the agent’s solvency, modifies the agent’s behavior. To determine the optimal consumption, the agent takes into account the energy stock but also his anticipations on the realizations of future shocks. In the fifties, different authors addressed the question of minimizing dispatch cost in a hydrothermal system. Little (1955) determined the optimal water management in an uncertain setting close to the one I use, but he did not focus on the way an electric park is valued. Two years later, Koopmans (1957) developed a model with two energy sources without uncertainty and aimed at determining the optimal water storage policy that minimizes the operating cost of thermal generation. In a second step, he tried to obtain the value of the power generated and of the water used and/or stored.

A parallel can be drawn between a dam that contains water and the savings of an agent and between the water flow that enters a dam and the random revenue of the agent. In models focusing on agents’ consumption/saving behavior, agents are usually assumed not being allowed to borrow at each time period. Without such liquidity constraints, agents would perfectly smooth their consumption over time. But with liquidity constraints, agents are not able any more to use an anticipated increase in their revenue in the future by increasing today the amount they are allowed to borrow. The introduction of such constraints decreases thus consumption even if they are not binding. Agents are indeed afraid of not being able to borrow. Such models have been studied by Deaton (1991), Zeldes (1989), Carroll (1997) and Gollier (2001).
In this article, I aim at determining the optimal electric park that generates power with two energy sources. I differ from the initial Hotelling model since the exhaustible resource is potentially renewable as it is regularly replenished with an uncertain flow. The analysis concerns the optimal management of random stocks. I consider the optimal allocation between two energy sources as Crampes and Moreaux (2001). However, two main features have been added: not only do I consider an infinite horizon model, but I also introduce uncertainty on the water inflow. A social planner chooses the energy production at each period depending on the state of the system and his expectations on its evolution. He maximizes the discounted sum of the expected utility he gets from the use of different energy sources. In a first step, the optimal production flow when only hydroelectricity is available is analyzed. Hydroelectricity generation comes from water stored in a dam that is supplied with a random inflow. Unlike Ayong Le Kama (2004), I do not consider any solvency constraint since future water inflows are assumed to be always positive. However, I add a second constraint on the availability of the resource since it must be finite. Therefore, a second kind of “liquidity constraint” is introduced: not only is the social planner unable to produce electricity from water not yet fallen in the reservoir, but it is also not possible to store more water in the dam than its capacity. I find that the management of the dam allows electricity smoothing when the quantity $z$ of available water is in a given range $[z^*, z^{**}]$ that I determine. Indeed in this region, the social planner prefers cutting down on total consumption today to let enough water in the dam for the future. But when the quantity of available water is too low (lower than some threshold $z^*$), it is completely consumed since the social planner expects future rainfalls will replenish the dam. A second energy source is then added and the optimal combination between the two energy sources allows for a better smoothing of electricity consumption even when the dam is full and the costly energy source is not consumed. Moreover the introduction of the second energy source shifts up water production. Besides the analysis of the allocation between the two sources, I consider the effect of an increase in uncertainty of the water inflow. For some values of the quantity of available water, more water is consumed under uncertainty than under certainty. Once the optimal allocation has been determined, I consider a long term situation where the characteristics of the electric park have to be determined. I compute the optimal number of dams and find that it is an increasing function of the price of the alternative energy source. Lastly, as an extension, I focus on the efficiency of time diversification when a second random energy source is introduced that is non-storable.

In the next section of this article, the model is presented. A benchmark case is studied in section 3 when the water inflow is constant. Section 4 is devoted to the resolution of the model in a general setting. Section 5 deals with the characteristics of an optimal electric park in the long term. In section 6, I propose as an extension to introduce a third energy source that is uncertain and non-storable. Section 7 concludes.

2 The Model

I consider a small economy in which a social planner produces electricity using two different technologies: hydroelectric energy and thermal power. Hydroelectricity is obtained from water extracted from a dam. The dam is supplied with a random water flow $\tilde{y}_t$ and is characterized by its capacity $\bar{Z}$. Thermal power is available at any time at a constant exogenous market
In this setting, hydroelectricity is a renewable resource whereas the relative scarcity of thermal power is expressed in its price: there is no constraint on its availability given its market price. Therefore, thermal power is a backstop technology of hydroelectricity in the sense used by Nordhaus (1979) and Heal (1993): “a technology that can provide substitutes for the resource once it is fully depleted, and can provide these substitutes on a very large scale indeed”.

I consider a setting where a social planner chooses at each period the energy production depending on the state of the system and his expectations on its evolution. He aims at maximizing the expected intertemporal utility he gets from the flow of future production. The use of the dam introduces constraints. On the one hand, consumption is restricted by the quantity of stored water in the dam, but on the other hand, water consumption has to be high enough since the quantity of stored water is bounded by the dam capacity $Z$. As for thermal power, the only constraint is a non-negativity one. I introduce

- $w_t$ the amount of water in the dam at the beginning of period $t$,
- $\tilde{y}_t$ the random flow of water that enters the reservoir,
- $z_t$ the total amount of water that is available to the agent in period $t$, that is $z_t = w_t + \tilde{y}_t$. This implies that $z_t \geq \min \tilde{y} \ \forall t$,
- $c_t$ the amount of water that is extracted from the dam in period $t$,
- $Z$ the dam capacity. $w_t$, $z_t$, $c_t$ and $Z$ are measured in cubic meters and refer to volumes of water in the reservoir. I assume $\min \tilde{y} < Z$,
- $x_t$ the quantity of electricity produced with thermal power consumed in period $t$ and measured in kWh,
- $p$ the unit price of thermal power,
- $R$ the conversion coefficient of a water volume into a quantity of produced electricity in kWh. It depends on the characteristics of the dam (its height, its flow),
- $\beta$ the discount factor with $\beta < 1$,
- $u$ the utility function: it is strictly increasing and strictly concave.

The timing is represented in Figure 1.

The dynamics of water available to the social planner satisfies $z_{t+1} = z_t - c_t + \tilde{y}_{t+1}$. Water consumption is bounded by the quantity of available water, Therefore, the first constraint reduces to $c_t \leq z_t$. Finally, the remaining stock of water should not exceed the dam capacity: $w_{t+1} \leq Z$. Taking into account the water dynamics, the constraint comes down to $z_t - c_t \leq Z$. The social planner’s program is

$$\max_{\{c_t,x_t\}_{t \geq 0}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t [u (Rc_t + x_t) - px_t]$$

(1)

\footnote{In this setting of a competitive market for thermal electricity, the price corresponds to the marginal cost.}
subject to
\[ z_{t+1} = z_t - c_t + \tilde{y}_{t+1}, \]  \( t \) \hspace{1cm} (2) 
\[ c_t \leq z_t, \] \hspace{1cm} (3) 
\[ c_t \geq z_t - \overline{z}, \] \hspace{1cm} (4) 
\[ c_t \geq 0, \] \hspace{1cm} (5) 
\[ x_t \geq 0, \] \hspace{1cm} (6) 
\[ z_0 \text{ given}. \] \hspace{1cm} (7)

I choose a CRRA utility function where the concavity coefficient \( \gamma \) equals the inverse of the constant price elasticity of the demand for thermal power \( x(p) \) taken in absolute value
\[ u(x) = \frac{x^{1-\gamma}}{1-\gamma}. \]

I assume moreover that \( \gamma > 1. \) Before solving the model, a first result on the shape of both consumption flows follows.

**Proposition 1** Thermal power is consumed after having consumed all available water in the reservoir: \( x_t > 0 \Rightarrow c_t = z_t. \)

The social planner aims at maximizing total electricity production at the lowest possible cost. Hydroelectricity production is limited by the dam’s capacity and has a random availability, therefore thermal electricity, even if it is costly, allows to soften these features. The result proved in Proposition 1 means that the electricity consumption path can be decomposed into two phases. The cheapest energy source is consumed first. Once the water reserve is fully depleted, thermal power is produced in combination with the water inflow. The driving force for this result is the willingness of the social planner to postpone energy expenditure because \( \beta < 1. \)

**Proof:** Suppose the results does not hold: \( x_t > 0 \) and \( c_t < z_t. \) Let \( t' \) be the first time period for which \( c_{t'} > 0 \) (it exists else constraint (4) would be violated). Consider the following strategy

- \( \{\tilde{c}_t, \tilde{x}_t\} \) with \( \tilde{c}_t = c_t + \frac{\varepsilon}{\tilde{R}} \) and \( \tilde{x}_t = x_t - \varepsilon, \)
- \( \{\tilde{c}_{t+1}, \tilde{x}_{t+1}\} \) with \( \tilde{c}_{t'} = c_{t'} - \frac{\varepsilon}{\tilde{R}} \) and \( \tilde{x}_{t'} = x_{t'} + \varepsilon. \)

\[ \Delta u_t = u_{t+1} - u_t = p \text{ and } \Delta u_{t'} = u_{t'+1} - u_{t'} = -p. \] Therefore, the total effect on the whole time horizon, \( \Delta u = \beta' p - \beta'' p = \beta' p \left(1 - \beta^{t-t'}\right) \) is strictly positive, and strategy \( \{\tilde{c}, \tilde{x}\} \) is strictly preferred to strategy \( \{c, x\} \) that is the optimal one. This leads to a contradiction. \( \square \)
In the light of this result, to solve the initial maximization program, I first determine the optimal consumption of thermal power in the second stage of the process, that is, when the dam is empty. I then use this information to determine the optimal consumption of hydroelectricity in the first stage. To do so, I introduce function
\[
\hat{u}(Rc; p) = \max_{x \geq 0} u(Rc + x) - px. \tag{8}
\]

**Lemma 1** Let \( e^*(p) = u'^{-1}(p) \) be the demand for electricity when thermal electricity is the unique energy source. Function \( \hat{u} \) is equal to
\[
\hat{u}(Rc; p) = \begin{cases} 
  u(e^*(p)) - p(e^*(p) - Rc) & \text{if } Rc \leq e^*(p), \\
  u(Rc) & \text{if } Rc \geq e^*(p).
\end{cases}
\]
Thus
\[
x(t) = \max (e^*(p) - Rc(t), 0),
\]
and demand for electricity is higher than or equal to \( e^*(p) \).

**Proof:** If \( \nu \) is the Lagrangian multiplier associated to the constraint \( x \geq 0 \), the FOC reads
\[u'(Rc + x) - p + \nu = 0.\]

With \( e^* = u'^{-1}(p) \), two cases occur:
- either \( \nu = 0 \), what implies \( x \geq 0 \). It follows that \( u'(Rc + x) = p = u'(e^*) \), \( x = e^* - Rc \geq 0 \), and \( Rc \leq e^* \),
- or \( \nu > 0 \), what implies \( x = 0 \) and \( u'(Rc + x) = p - \nu < u'(e^*) \), \( x = 0 \), and \( Rc > e^* \).

The shape of \( \hat{u} \) is straightforward. \( \Box \)

\( \hat{u} \) is represented in Figure 2. When \( c \leq e^*(p)/R \), it is a straight line and once this threshold is crossed, it equals \( u \). In order the problem to be meaningful, I assume \( e^*(p)/R < Z \).

![Figure 2: Indirect utility function \( \hat{u} \)](image)

The social planner's program reduces to
\[
\max_{\{c_t\}_{t \geq 0}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \hat{u}(Rc_t) \tag{9}
\]
subject to

\[ z_{t+1} = z_t - c_t + \tilde{y}_{t+1}, \]  
(10)

\[ c_t \leq z_t, \]  
(11)

\[ c_t \geq z_t - Z, \]  
(12)

\[ c_t \geq 0, \]  
(13)

\[ z_0 \text{ given.} \]  
(14)

Before I characterize the solution of this program, I study, as a benchmark, the case without uncertainty on the water inflow.

3 Benchmark: model under certainty

The water inflow that fills the dam at each period is assumed to be constant: \( \forall t, \tilde{y}_t = y > 0 \). Total consumption is at each period greater or equal than \( y \): \( \forall t, c_t \geq y \) and constraint (13) is thus always satisfied.

Lemma 2 Without uncertainty on the water inflow, water consumption is decreasing until the dam is empty. Afterwards, it equals the water inflow. Thermal electricity is consumed if and only if the water inflow is not sufficient to satisfy demand \( e^*(p) \).

The different cases are represented in Table 1.

<table>
<thead>
<tr>
<th>( y &gt; e^*(p)/R )</th>
<th>electricity (from water)</th>
<th>electricity (from thermal power)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y \leq e^<em>(p)/R ) ( \text{and } z_0 \geq e^</em>(p)/R )</td>
<td>( x = 0 )</td>
<td>( c ) is decreasing until ( y ) and constant afterwards</td>
</tr>
<tr>
<td>( y \leq e^<em>(p)/R ) ( \text{and } z_0 &lt; e^</em>(p)/R )</td>
<td>( x ) is strictly positive ( \text{once } c \text{ reaches } e^*(p)/R )</td>
<td>( c ) is decreasing until ( y ) and constant afterwards</td>
</tr>
</tbody>
</table>

Table 1: Electricity consumption paths

**Proof:** According to Lemma 1, at each time period, electricity consumption is at least equal to \( e^*(p) \). Two cases may occur:

- either \( y < z_t \leq e^*(p) R \): there remains a strictly positive quantity of water in the dam before the precipitation replenishes the reservoir \( (w_t > 0) \),

- or \( z_t = y \leq e^*(p)/R \): there is no water stock in the dam any more and at each time period, water consumption is equal to the rainfall. \( x(t) + Rc(t) > e^*(p) \) implies that no thermal power is consumed \( (x(t) = 0) \) and electricity is produced using only the hydroelectric technology.

Optimal thermal power consumption path is therefore equal to \( x(t) = \max(e^*(p) - Rc(t), 0) \).

In this case, one of the limit of hydroelectricity, its random availability, is eliminated, and the unique constraint consists in its stock capacity. However, this stock limit only applies when the
initial quantity of water in the dam is very high (when \( z_0 > \overline{Z} - y \)), and in this case, the social planner consumes all the surplus in the first time period and the water inflow afterwards. In the certain case as in the general case, thermal electricity is consumed when the dam is empty. But in the certain case, \( c \) is a decreasing function of time. The resolution of the program is in Appendix A. I present the shape of the optimal water consumption path in Figure 3. \( \gamma \) is taken to be equal to 5. This is consistent with the estimation of the price elasticity for different European countries found in Söderholm (2001) whose mean amounts to -0.2. The constant inflow of water, \( y \), is equal to 2. Concerning the other parameters, the values chosen are \( \beta = 0.95, R = 0.7, \overline{Z} = 10, z_0 = 10 \) and \( p = 0.05 \). \( e^*(p) \) is therefore equal to 1.82.

![Figure 3: Optimal power consumption path](image)

In order to be able to draw comparisons with the case where the water inflow is uncertain (see following section), I give the shape of the water consumption flow \( c \) relative to the quantity of available water \( z \) in Figure 4.

![Figure 4: Optimal water consumption path as a function of the quantity of water available](image)
There is a kink: indeed, for low levels of stored water, all the available water is consumed. Afterwards, this is a step function because of the definition of the time $T$ from which thermal power is consumed. As I work in discrete time, $T$ has to be integer. I focus now on the general case.

4 General case: model with uncertainty

I assume now that each realization of the random variable $\tilde{y}_t$ is positive meaning that $z_t > \min \tilde{y}$, $\forall t$. The resolution of problem (9) is made more convenient by using the Bellman equation

$$v(z) = \max_c \{ \tilde{u}(Rc) + \beta \mathbb{E}v(z - c + \tilde{y}) \}$$

subject to

$$c \leq z, \quad (16)$$

$$c \geq z - Z, \quad (17)$$

$$c \geq 0. \quad (18)$$

The following lemma gives a first result on the shape of the value function $v$.

**Lemma 3** The value function $v$ is concave.

**Proof:** See the Appendix.

This technical result is a first step before obtaining results on the shape of both consumption flows (see the following subsection). The FOC reduces to

$$R\tilde{u}'(Rc) \begin{cases} \geq \beta \mathbb{E}u'(z + \tilde{y} - c) & \text{if (16) is binding}, \\ \leq \beta \mathbb{E}u'(z + \tilde{y} - c) & \text{if (17) or (18) is binding}, \\ = \beta \mathbb{E}u'(z + \tilde{y} - c) & \text{otherwise}. \end{cases}$$

The second order condition, $\frac{\partial^2 L}{\partial c^2} = R^2\tilde{u}''(Rc) + \beta \mathbb{E}v''(z + \tilde{y} - c) \leq 0$, is satisfied because of the concavity of $u$ and $v$.

A temporal study of the consumption flows is not possible in the uncertain case. Therefore, the analysis is conducted focusing on the state variable $z$. The realization of $z_{t+1}$ knowing $z_t$ is of course a random variable dependent on the realization of the random water inflow. There are two means for the social planner to smooth electricity consumption: consuming thermal power when water extraction is low or storing water in the dam when precipitation is large. According to Proposition 1, when water is scarce, the social planner does not use the dam to store water. He prefers consuming all the water available and smoothing electricity consumption with the consumption of thermal power. On the contrary, when there is more water in the dam, the social planner does not use the alternative energy source anymore, but the dam to smooth electricity consumption.

4.1 Analysis of the water consumption flow

I begin this section with a result on the shape of function $c$.

**Lemma 4** Electricity consumption is strictly positive when the quantity of available water is strictly positive: $\forall z > 0, c(z) > 0$. 

10
The numerical resolution of problem (15) is represented in Figure 5. I consider a random inflow $\tilde{y}$ that takes the values 0, 3 and 6 with equal probabilities, thus $\forall t, z_t \geq 0$.

There are two thresholds $z^*$ and $z^{**}$ such that

- $\forall z \leq z^*$, $c(z) = z$,
- $\forall z \geq z^{**}$, $c(z) = z - Z^3$.

**Proposition 2** The following property holds:

$$z^* \geq e^*(p) / R.$$  \hfill (19)

**Proof:** Suppose this is not the case and let $\bar{z} \in [z^*, e^*(p) / R[$:

- since $\bar{z} > z^*$, $c(z) < z$ and there remains water in the reservoir,
- since $\bar{z} < e^*(p) / R$, according to Lemma 1, $x(\bar{z}) = e^*(p) / R - c(\bar{z}) > e^*(p) / R - \bar{z} > 0$.

This implies that thermal power is consumed whereas there remains water in the reservoir. This contradicts Proposition 1. \hfill $\Box$

Proposition 2 implies that

- $\forall z \leq \frac{e^*}{R}$, $c(z) = z$, and $x(z) = e^*(p) - Rz$,
- $\forall z \in \left[\frac{e^*}{R}, z^*\right]$, $c(z) = z$, and $x(z) = 0$,
- $\forall z \geq z^*$, $x(z) = 0$, and $c(z)$ is the solution of $Rz' (Rc) = \beta E' (z - c + \tilde{y}) + \lambda - \eta$.

$^3$The existence of $z^*$ has already been proven by Deaton (1991) and Deaton and Laroque (1992, 1996). In Deaton's model studying the consumption/saving behavior of agents, there is only one constraint on the maximal amount that can be borrowed which corresponds to (16).
where $\lambda$ and $\eta$ are the Lagrange multipliers associated with constraints (16) and (17).\(^4\)

In order to understand how hydroelectricity consumption smoothing is possible for different levels of stored water, I compute the marginal propensity to consume $\partial c/\partial z$ from the first order condition. It is a measure of the efficiency of intertemporal smoothing going from 0 in case of a perfect smoothing to 1 if water is never stored.

$$\frac{\partial c}{\partial z} = \begin{cases} 
1 & \text{if } z \leq z^*, \\
\frac{\beta E v''(z+\tilde{y}-c(z))}{R^2 v''(Rc(z))+\beta E v''(z+\tilde{y}-c(z))} & \text{if } z^* < z < z^{**}, \\
1 & \text{if } z \geq z^{**}.
\end{cases}$$

First, $c'(z)$ is positive implying that $c$ is an increasing function. Next, note that when neither (16) nor (17) is binding, the marginal propensity to consume is strictly less than 1 and time diversification is possible:

- when $z < z^*$, the reservoir is emptied out. Either thermal power is consumed or, when $z \in [e^* (p)/R, Z^*]$, the social planner knows that at the next time period there will be a water inflow greater or equal then $\min \tilde{y}$, there is thus no need to keep water in stock,

- when $z \in [z^*, z^{**}]$, as the social planner knows that in the future he could not produce the quantity of water he would like, he prefers cutting down on production today to let enough water in the dam for the future,\(^5\)

- when $z \geq z^{**}$, the social planner increases hydroelectricity production because of the risk of a very rainy period for many successive periods.

Note that the marginal propensity to consume is first decreasing and then increasing. Therefore, function $c$ is successively concave then convex.\(^6\)

It is also meaningful to look at the evolution of hydroelectricity production over time. In Figure 6, the consumption flow for 100 time periods is represented together with the water stock. It has been obtained with a simulation of the random variable $\tilde{y}$ (in this numerical illustration, I consider the extreme case where $p \to +\infty$, meaning that thermal energy is not available).

The path of water consumption has a completely different shape than under certainty where it was a decreasing function of time. Observe that the variations in the consumption flow are smaller than the stock variations: consumption smoothing is efficient.

4.2 Description of the thermal energy consumption flow

Once the water consumption flow is known, thermal power consumption is equal $x(z) = \max(e^*(p) - Rz, 0)$ (see Lemma 1 and Proposition 2). The production of thermal power decreases linearly with $z$ from $e^*(p)$ down to 0.

4.3 Comparative static relative to price

The following lemma provides a first result on the evolution of thermal power consumption with respect to price.

\(^4\)Lemma 4 tells that constraint $c \geq 0$ in program (15) is never binding when $z > 0$.

\(^5\)In the case where $p \to +\infty$ (the alternative energy source is not available), it can be shown that $z^* \geq \min \tilde{y}$ (see Appendix C).

\(^6\)The difference with the concavity result of the consumption function proven by Carroll and Kimball (1996) comes from the second constraint: consumption has to be high enough in our model.
Lemma 5 For a given level of available water in the dam $z$, thermal electricity production is decreasing with $p$.

Proof: It is straightforward knowing that $e^*(p)$ is a decreasing function of $p$ and that $x(z,p) = \max (e^*(p) - Rz, 0)$.

The numerical resolution reveals that $c(z;p)$ is decreasing with $p$: for a given level of water in the reservoir, when the price of the alternative electricity source increases, hydroelectricity production decreases. This result could appear as counterintuitive since the social planner does not take advantage of the cost decrease of the thermal technology to increase his water consumption. On the contrary, he even decreases it. Remember that this is a dynamic problem and that the main concern is to be able to produce enough electricity to face a potential unfavorable future. But when $p$ increases, one means to smooth electricity consumption is less efficient. Indeed, when $p$ increases, the social planner knows that thermal power production is going to
decrease (see Lemma 5). That is why, for a given value of \( z \), he prefers decreasing hydroelectricity production in order to keep water stock for the future. This shift of the consumption flow when the price of the second energy source increases expresses a precautionary behavior of the social planner. Moreover note that as \( p \) decreases, \( \partial c/\partial z \) decreases: the existence of thermal power at a low price improves intertemporal diversification even when the fossil source is not consumed (for values of \( z \) such that \( e^*(p) - Rz = 0 \)).

Table 2 presents the proportion of thermal power and water in the total amount of electricity consumed for different values for \( p \). These values have been obtained by simulating the random variable \( \tilde{y} \) 10000 times: one obtains a path for the water stock for 10000 periods and consequently both consumption flows.

<table>
<thead>
<tr>
<th>( p )</th>
<th>electricity (from water)</th>
<th>electricity (from thermal power)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>98.51%</td>
<td>1.49%</td>
</tr>
<tr>
<td>0.075</td>
<td>97.73%</td>
<td>2.27%</td>
</tr>
<tr>
<td>0.05</td>
<td>95.89%</td>
<td>4.11%</td>
</tr>
<tr>
<td>0.025</td>
<td>92.42%</td>
<td>7.58%</td>
</tr>
<tr>
<td>0.01</td>
<td>86.77%</td>
<td>13.23%</td>
</tr>
</tbody>
</table>

Table 2: Proportion of hydroelectricity and thermal power in the total consumption for different values for \( p \)

As \( p \) decreases, the proportion of thermal power increases and the proportion of hydroelectricity decreases. This happens in an exponential way. This result completes the result on the precautionary behavior of the social planner developed at the beginning of the subsection. When \( p \) increases, although, for a given quantity \( z \) of available water, the social planner reduces hydroelectricity production to keep water in stock for the future, the share of hydroelectricity production relative to thermal power production increases.

To conclude this paragraph on the comparative statics with respect to price, I state the following result

**Lemma 6** The value function \( v \) decreases with \( p \).

**Proof:** See the Appendix.

This result highlights the link between the two energy sources: when the price of thermal electricity increases, the dam is more valuable and \( v \) increases.

### 4.4 Comparative static relative to uncertainty

To study the impact of an increase in uncertainty of the water inflow on electricity production, I write the random inflow as the sum of a constant random term and a zero mean random variable: \( \tilde{y} = \bar{y} + \tilde{\varepsilon} \). I consider three cases: one without uncertainty and the other with two random variables \( \tilde{\varepsilon}_1 \) and \( \tilde{\varepsilon}_2 \) such that

\[
\sigma_1^2 = \text{Var}(\tilde{\varepsilon}_1) < \sigma_2^2 = \text{Var}(\tilde{\varepsilon}_2).
\]  

In Figure 8, the three consumption flows are represented for value of \( z \) close to \( z^* \).

When \( z \) is close to \( z^* \), the more uncertainty, the higher the consumption level. The social planner could have decided to stock more water in order to face the more uncertain future. But
when he is close to empty out the reservoir, he does not reduce his consumption, meaning that he does not adopt a prudent behavior. According to Leland (1968), in a finite horizon setting, an agent is prudent (in the sense where he consumes less today and saves more) if and only if the marginal utility of future consumption is convex. The following lemma adapts this result to this context in the special case where there are two random variables \( \tilde{\epsilon}_1 \) and \( \tilde{\epsilon}_2 \) such that \( \tilde{\epsilon}_1 = \{-\varepsilon, \varepsilon; 0.5, 0.5\} \) and \( \tilde{\epsilon}_2 = \{-\varepsilon, \varepsilon; 0.5, 0.5\} \) with \( \varepsilon < \bar{\varepsilon} \).

**Lemma 7** The social planner is prudent if and only if

\[
\Delta \varepsilon \left( \frac{\partial^2 v(z - c_2 + \bar{y} + \varepsilon, \sigma_2)}{\partial z^2} - \frac{\partial^2 v(z - c_2 + \bar{y} - \varepsilon, \sigma_2)}{\partial z^2} \right) > -\Delta \sigma \left( \frac{\partial^2 v(z - c_2 + \bar{y} + \varepsilon, \sigma_2)}{\partial \sigma \partial z} + \frac{\partial^2 v(z - c_2 + \bar{y} - \varepsilon, \sigma_2)}{\partial \sigma \partial z} \right). 
\]

(21)

**Proof:** See the Appendix.

When the social planner is prudent, the RHS of (21) is negative. Therefore, the condition of the lemma is satisfied when \( v' \) is convex (the LHS of (21) is positive) or when \( v' \) is not too concave. But no general result holds that links the shape of \( v' \) and the prudence of the social planner. When \( z \) is close to \( z^* \), \( \hat{u}' \) is locally concave implying that \( v' \) is locally concave too (see Figure 8).\(^7\) Inequality (21) is not satisfied in this case and the social planner does not adopt a prudent behavior. What is the intuition for the result? Imagine the system is such that the quantity of water at the beginning of the period is close to \( z^* \). According to the optimal consumption rule, before rainfalls fill the dam, it is almost empty. Two realizations of the random variable may occur:

- **the bad state of nature occurs.**
  The rainfalls are too low to reach a hydroelectricity consumption level equal to \( e^*(p) / R \). The dam is thus totally emptied out. There is no precautionary saving in this case, and the increase in uncertainty does not influence the consumption policy of the social planner.

- **the good state of nature occurs.**
  The water stock is thus higher in the case of higher uncertainty and water consumption is also higher.

\(^7\)Note that the envelope theorem implies that \( v'(z) = R\hat{u}'(c(z)) \).
The increase in uncertainty is thus rather a good news when the system is such that \( z \) is close to \( z^* \) and the social planner increases thus consumption. But this does not hold for all the possible states \( z \). Indeed for other values of \( z \), \( v' \) is convex and equation (21) is satisfied, meaning that the social planner prevents from uncertainty by reducing his consumption. Moreover the numerical simulation reveals that the value function decreases with uncertainty.

Note that this result on the social planner’s prudence is close to the one obtained by Hoel [11]. Indeed he studied the optimal exhaustible resource extraction when the future substitute has an uncertain cost. He found that if the marginal utility of future consumption is concave, an increase in cost’s uncertainty leads to an increase in the extraction rate.

I now turn to the analysis of the optimal size of an electric park composed of two energy sources.

5 What is the optimal infrastructure?

I first introduce the number of dams as a parameter in program (15)\(^8\)

\[
v(z; \alpha, p) = \max_{c, x} \left\{ u(Rc + x) - px + \beta \mathbb{E}v(z - c + \alpha \tilde{y}; \alpha, p) \right\}
\]

subject to

\[
c \leq z,
\]

\[
c \geq z - \alpha Z,
\]

\[
x \geq 0.
\]

Variable \( \alpha \) stands for the number of dams. If \( \alpha \) dams are used, the total flow of water that fills then amounts to \( \alpha \tilde{y} \) and the total capacity is equal to \( \alpha Z \). \( \tilde{y} \) and \( Z \) are exogenous parameters. As \( \alpha \) increases, two effects appear:

- as \( \alpha \) increases, it is as if the dam’s capacity increased. Consumption smoothing should thus be more efficient leading to an increase in the value of the electric park.
- as \( \alpha \) increases, the quantity of rainfalls that fills the dams increases also. The value of the electric park also increases.

Figure 9 confirms this intuition: \( v \) increases with \( \alpha \). Moreover, as \( p \) increases, \( v(0; \alpha, p) \) increases: the two goods, dams and thermal electricity, are substitutes indeed. I also note that \( \frac{\partial v(0; \alpha, p)}{\partial \alpha} \) is a decreasing function of \( p \): the dam’s marginal value increases with the cost of thermal electricity.

To obtain the optimal number of dams, the social planner solves

\[
\max_{\alpha \geq 0} v(0; \alpha, p) - C(\alpha)
\]

subject to the participation constraint

\[
v(0; \alpha, p) - C(\alpha) \geq \frac{1}{\beta - 1} \left( u \left( u'^{-1}(p) \right) - pu'^{-1}(p) \right),
\]

where \( \frac{1}{\beta - 1} \left( u \left( u'^{-1}(p) \right) - pu'^{-1}(p) \right) \) is the utility without hydroelectricity.

With a linear cost function, the numerical resolution whose results are reported in Table 3 reveals that \( \alpha^*(p) \) is an increasing function of \( p \).

\(^8\)I choose to focus on the optimal number of dams although other interpretations would have been possible (optimal size of the dams...).
When thermal power price increases, it is optimal to build more dams. However, when $p$ is low and thus thermal electricity quite cheap, the participation constraint is not satisfied anymore. Hydroelectricity is not used anymore since the fixed costs are too high. Note that when uncertainty increases, the number of dams also increases although $v$ decreases with uncertainty (see the previous section). This may be due to the capacity increases it provides.

### 6 Extensions

In this part, I suppose that a third energy source is available, wind energy for instance. I focus on hydro power and wind power. Indeed, I know from section 2 that introducing thermal power will be equivalent to using utility function $\hat{u}$ instead of utility function $u$.

Suppose the social planner can provide hydroelectricity and wind electricity. The dam is supplied with a random inflow. Wind energy is also random. Two cases occur: either there is no wind, in which case no wind electricity can be consumed (which happens with probability $q$), or there is wind and the quantity of wind electricity amounts to $W$ (which happens with probability $(1 - q)$). Let $c_0$ (respectively $c_1$) be the water consumption flow when wind electricity is available (respectively not available) and $v_0(z)$ (respectively $v_1(z)$) be the value function when wind electricity is available (respectively not available). The social planner’s program is then the following
\[
\begin{align*}
\begin{cases}
v_0 (z) &= \max_{c_0} u(Rc_0 + \beta [q\mathbb{E} v_0 (z - c_0 + \tilde{y}) + (1 - q) \mathbb{E} v_1 (z - c_0 + \tilde{y})]) \\
\text{subject to } c_0 \leq z, c_0 \geq 0, c_0 \geq z - \bar{Z},
\end{cases} \\
v_1 (z) &= \max_{c_1} u(Rc_1 + \bar{W}) + \beta [q\mathbb{E} v_0 (z - c_1 + \tilde{y}) + (1 - q) \mathbb{E} v_1 (z - c_1 + \tilde{y})] \\
\text{subject to } c_1 \leq z, c_1 \geq 0, c_1 \geq z - \bar{Z}.
\end{align*}
\]

I denote $\lambda_0$, $\mu_0$ and $\nu_0$ (resp. $\lambda_1$, $\mu_1$ and $\nu_1$) the three Lagrange multipliers associated with the constraints that apply to $c_0$ (resp. $c_1$). I first have some results on the shape of the consumption flows.

**Lemma 8** Consumption flows $c_0 (z)$ and $c_1 (z)$ have the following properties:

1. $\forall z > 0, c_0 (z) > 0$,
2. if $c_1 (z) = 0$, then $Rc_0 (z) < \bar{W}$,
3. if $c_1 (z) = z$, then $c_0 (z) = z$. The opposite is not true,
4. if $c_0 (z) = z - \bar{Z}$, then $c_1 (z) = z - \bar{Z}$. Once more, the opposite is not true.

**Proof:** See the Appendix.

Note that the constraint $c_0 (z) \geq 0$ can be omitted. When there is no wind, as soon as there is a strictly positive quantity of available water, it is consumed. Unlike the case without wind power, $c_1 (z)$ might be equal to zero even when the quantity of available water is strictly positive. Indeed, when there is wind power, the social planner is tempted to keep water in stock for the future in case wind power and/or precipitation are low the following time period. Thus, when $z$ is very low, he prefers not producing hydropower. However, this only happens for some quantity of available water $z$ such that if there were no wind, the total quantity of hydroelectricity produced would be less than the quantity of wind electricity $\bar{W}$. This means that, when the quantity of available water is high enough, water is consumed whatever the quantity of wind power. The last two results represent a first step completed with the following proposition. They allow a preliminary ranking of both consumption flows $c_0$ and $c_1$ when one of the constraints is binding.

**Proposition 3** For any level of available water in the dam $z$:

1. hydroelectricity consumption is higher when there is no wind: $c_1 (z) \leq c_0 (z)$,
2. electricity consumption is higher when there is wind: $Rc_0 (z) \leq Rc_1 (z) + \bar{W}$.

**Proof:** See the Appendix.

When wind power is available, the social planner prefers saving water and taking advantage of wind power. However, total electricity consumption is higher when there is wind power. The two consumption flows $c_0 (z)$ and $c_1 (z)$ are represented in Figure 10.

The introduction of this third energy source that is random but non-storable increases electricity consumption but does not produce a visible effect on time diversification. Indeed the slope of $c_0 (z)$ is not very different from the slope of $c_1 (z)$ (see Figure 10).
The introduction of a second energy source improves intertemporal smoothing. When the price of this alternative energy source decreases, the marginal propensity to consume electricity decreases, illustrating an improvement of the time diversification effect. This result even holds on ranges of the water stock for which hydroelectricity is the only energy source that is consumed. Indeed, with two energy sources, there exists a minimum level of electricity that is produced at each period and thermal power is used to reach this level when there is not enough water. But as soon as the quantity of water is sufficient to reach or to exceed this level, the social planner does not use the costly energy source any more and prefers consuming water exclusively. Moreover the presence of thermal power shifts up the consumption flow. Indeed, when the price of thermal power increases, the social planner adopts a precautionary behavior. He prefers producing less to constitute a greater stock for the future.

The optimal capacity of the total infrastructure is increasing with the thermal power price. The less thermal power is produced (what occurs when \( p \) is high), the more dams are needed to smooth consumption.
The introduction of a second uncertain energy source that is not storable allows to increase hydroelectricity consumption for a given quantity of available water. However, water consumption smoothing is not significantly increased since the new energy source is uncertain. It is the certain availability of thermal energy that allows to improve time diversification.

References


A Computation of the optimal consumption flows in the certain case (Section 2)

I present the computations in two cases.

**Case 1:** \( y \leq e^* (p) / R \) and \( z_0 \geq e^* (p) / R \):

Denoting by \( T \) the last time period for which no thermal power is consumed, the program of the social planner reads

\[
\max_{c(t)} \sum_{t=0}^{T} \beta^t u(Rc(t)) + \beta^{T+1} [u(e^* (p)) - p(e^* (p) - Rc(T + 1))] - \sum_{t=T+2}^{\infty} \beta^t (u(e^* (p))) - p(e^* (p) - Ry))
\]

subject to

\[
\begin{align*}
    z_{t+1} &= z_t - c_t + y, \\
    c(T) &> \frac{e^* (p)}{R}, \\
    c(T + 1) &= z_0 + (T + 1) y - \sum_{t=0}^{t=T} c_t, \\
    c(T + 1) &\leq \frac{e^* (p)}{R}, \\
    c(T + 1) &\geq y, \\
    z_0 \text{ given.}
\end{align*}
\]

The first constraint is the usual one on the dynamics of available water, the second inequality means that at time \( T \), no thermal power is consumed yet. The third equation means that at time \( T + 1 \), the dam is completely emptied out. Indeed, if it was not the case, water would remain in the dam in period \( T + 1 \). Therefore, no thermal power would be consumed in this time period what would not be consistent with the definition of \( T \). The fourth inequality means that at time \( T + 1 \), thermal power is consumed
and the last equality ensures that until $T$, no more water than the quantity stored in the dam has been consumed.

The resolution of this program leads to the following results:

$$c(t) = \begin{cases} 
\beta^{\frac{T-t}{T+1}} c(T) & \text{if } t \leq T, \\
 z_0 + (T+1) y - c(T) \frac{\beta^{\frac{T-t}{T+1}} - 1}{1 - \beta^{\frac{T-t}{T+1}}} & \text{if } t = T+1, \\
y & \text{if } t \geq T+2,
\end{cases}$$

where $T$ is the lowest integer $t$ such that

$$z_0 + (t+1) y < \frac{c^*(p)}{R}.$$

and where $c(T)$ is given by the following expression

$$c(T) = \begin{cases} 
\frac{c^*(p) \beta^{\frac{1}{2}}}{1 - \beta^{\frac{1}{2}}} & \text{if } z_0 + (T+1) y - \frac{c^*(p) \beta^{\frac{T+1}{1 - \beta^{\frac{T+1}{T+1}}}} < \frac{c^*(p)}{R}}, \\
\frac{(z_0 + Ty)(1 - \beta^{\frac{1}{2}})}{\beta^{\frac{1}{2}} - \beta^{\frac{1}{2}}} & \text{else}.
\end{cases}$$

**Case 2: $y > \frac{c^*(p)}{R}$**

In this case

$$c(t) = \begin{cases} 
\beta^{\frac{T-t}{T+1}} c(T) & \text{if } t \leq T, \\
y & \text{if } t \geq T+1,
\end{cases}$$

where $T$ is the highest integer $t$ such that

$$\frac{(z_0 + ty)(1 - \beta^{\frac{1}{2}})}{\beta^{\frac{1}{2}} - \beta^{\frac{1}{2}}} > y,$$

and where $c(T)$ is given by the following expression

$$c(T) = \frac{(z_0 + Ty)(1 - \beta^{\frac{1}{2}})}{\beta^{\frac{1}{2}} - \beta^{\frac{1}{2}}}.$$

**B Computation of the value function $v$ in the certain case (Section 2)**

I begin with the most interesting case when $\frac{c^*(p)}{R} > y$.

The water consumption path has the following expression (as I already noted in section 2, this amounts to studying $c_0$ with respect to $z_0$):

$$c(z) = \max \left( \beta^{-\frac{1}{2}} c(T), z - Z \right) \text{ if } z > \frac{c^*(p)}{R} \beta^{-\frac{1}{4}},$$

else.

The shape of the value function depends on the initial quantity of water. Three cases have to be distinguished.

1. $z < \frac{c^*(p)}{R}$

   In this case, in the first period, all the water is consumed and thermal power is consumed in order to reach $c^*(p)$. In the following periods, water is consumed in quantity $y$ and thermal power in quantity $c^*(p) - Ry$. Therefore, the value function takes the following form:

   $$v(z) = u(c^*(p) - p(c^*(p) - Ry) + \sum_{t=1}^{\infty} \beta^t (u(c^*(p)) - p(c^*(p) - Ry))$$

   $$= \frac{(c^*(p))^{1-\gamma}}{1 - \gamma} - p(c^*(p) - Ry) + \frac{\beta}{1 - \beta} \left( \frac{(c^*(p))^{1-\gamma}}{1 - \gamma} - p(c^*(p) - Ry) \right).$$
2. $\frac{e^*(p)}{R} < z < e^*(p) \beta^{-\frac{1}{\gamma}}$.

The only difference with the previous case is that no thermal power is consumed in the initial period because there is enough water:

\[
v(z) = u(Rz) + \sum_{t=1}^{+\infty} \beta^t (u(e^*(p)) - p(e^*(p) - Ry))
\]

\[
= \frac{(Rz)^{1-\gamma}}{1-\gamma} + \frac{\beta}{1-\beta} \left( \frac{(e^*(p))^{1-\gamma}}{1-\gamma} - p(e^*(p) - Ry) \right).
\]

3. $z > e^*(p) \beta^{-\frac{1}{\gamma}}$.

In this case there are three periods:

- until period $T$, only water is consumed,
- in period $T + 1$, all the water that remains in the dam is consumed and thermal power is consumed to reach the level $e^*(p)$ of electricity,
- from period $T + 2$ on, there is no water in the dam any more. Only the consumption flow $y$ is consumed, therefore thermal power is consumed in quantity $e^*(p) - Ry$.

Therefore, the value function has the following expression:

\[
v(z) = \sum_{t=0}^{T} (\beta^t u(Rc_T)) + \beta^{T+1} (u(e^*(p)) - p(e^*(p) - Rc(T + 1))) + \sum_{t=T+2}^{+\infty} \left( (\beta^t (u(e^*(p)) - p(e^*(p) - Ry)) \right).
\]

It follows that:

\[
v(z) = \frac{(Rc(T))^{1-\gamma} \beta^{-\frac{1}{\gamma}}}{1-\gamma} \frac{1-\beta^{T+1}}{1-\beta} + \beta^{T+1} \left( \frac{(e^*(p))^{1-\gamma}}{1-\gamma} - p(e^*(p) - Rc(T + 1)) \right) + \\
+ \left( \frac{(e^*(p))^{1-\gamma}}{1-\gamma} - p(e^*(p) - Ry) \right) \frac{\beta^{T+2}}{1-\beta}.
\]

For sake of completeness, I now focus on the case where $\frac{e^*(p)}{R} < y$.

First of all, in this case:

\[
e(z) = \max\left( \frac{1 - \beta^\frac{1}{\gamma}(z + Ty)}{1 - \beta^\frac{1}{\gamma}}, z - Z \right)
\] if $z > y,$

else.

Following the same steps than above, I compute the value function for the different value taken by the initial level of water in the dam.

1. $z < \frac{e^*(p)}{R}$:

In this case, thermal power and water are consumed in the initial period, and from the second period on, water is only consumed in quantity $y$.

\[
v(z) = u(e^*(p)) - p(e^*(p) - Rz) + \sum_{t=1}^{+\infty} \beta^t (u(Ry))
\]

\[
= \frac{(e^*(p))^{1-\gamma}}{1-\gamma} - p(e^*(p) - Rz) + \frac{\beta}{1-\beta} \frac{(Ry)^{1-\gamma}}{1-\gamma}.
\]

2. $\frac{e^*(p)}{R} < z < y$:

Here also, the only difference with the previous case is that in the first period, no thermal power
is consumed. But, from the second period on, the consumption path is the same.

\[ v(z) = u(Rz) + \sum_{t=1}^{\infty} \beta^t u(Ry) = \frac{(Rz)^{1-\gamma}}{1-\gamma} + \beta \frac{(Ry)^{1-\gamma}}{1-\beta} \]

3. \( z > y \):
In this case, there exists a threshold \( T \) that defines the time from which on thermal power is consumed:

\[ v(z) = \sum_{t=0}^{T} (\beta^t u(RcT)) + \sum_{t=T+1}^{\infty} \beta^t u(Ry). \]

It follows that:

\[ v(z) = \frac{(Rc(T))^{1-\gamma}}{1-\gamma} \beta^{-\frac{T}{1-\gamma}} 1 - \beta \frac{T+1}{1-\gamma} + \beta^{T+1} ((Ry)^{1-\gamma}) \]

C Lemma

Lemma 9 When \( p \to +\infty \), the threshold \( z^* \) is higher than the minimum water inflow \( \min \tilde{y} \).

Proof: Suppose this result does not hold: \( \min \tilde{y} > z^* \). Denoting \( \bar{z} = \min \tilde{y} \) and \( \bar{c} = c(\bar{z}) \), this leads to \( \bar{c} < \bar{z} \). The FOC leads to

\[ Ru' (R\bar{c}) = \beta \bar{c} v' (\bar{c} + \tilde{y}) , \]

\[ < \beta v' (2\bar{c} - \bar{c}) , \]

\[ < \beta v' (\bar{z}) \text{ (since } \bar{c} < \bar{z} \text{) ,} \]

\[ < v' (\bar{z}) , \]

\[ = Ru' (R\bar{c}) \text{ (envelope theorem).} \]

Therefore, \( \min \tilde{y} \leq z^* \). \( \square \)

D Proofs

Proof of Lemma 3:
I apply Theorem 9.8 page 265 in Stokey and Lucas [19]. All the assumptions are satisfied:
1. \( X = [0, \bar{Z} + \max \tilde{y}] \) is a convex subset of \( \mathbb{R} \),
2. \( \tilde{y} \) is a discrete random variable that takes a finite number of values: \( \tilde{y} \in \{y_1, y_2, \ldots, y_n\} \),
3. the correspondence \( \Gamma : X \to X \) describing the feasibility constraints is non empty, compact valued and continuous (see Figure 11),
4. \( \hat{u}(x) \) is bounded and continuous, and \( \beta < 1 \),
5. \( \hat{u} \) is concave (\( \gamma < 1 \)),
6. \( \Gamma \) is convex (see Figure 11).

Thus, according to Theorem 9.8 p. 265 of Stokey and Lucas [19], function \( v \) is concave.

Proof of Lemma 4:
Suppose that there exists \( z_0 > 0 \) such that \( c(z_0) = 0 \). In this case, constraint (16) does not bind and the FOC reads

\[ Ru' (Rc(z_0)) \leq \beta \bar{c} v' (z_0 - c(z_0) + \tilde{y}) , \]

\[ = \beta \bar{c} v' (z_0 + \tilde{y}) , \]

\[ \leq \beta v' (z_0) \text{ because } v \text{ is concave,} \]

\[ < v' (z_0) \text{ because } \beta < 1 , \]

\[ = Ru' (Rc(z_0)) \text{ because of the envelope theorem.} \]
Therefore, there is a contradiction and \( c(z_0) > 0 \) (and eventually that constraint (16) binds leading to \( c(z_0) = z_0 > 0 \)).

**Proof of Lemma 6:**

Consider \( p_1 < p_0 \) and recall that

\[
v(z_t; p) = \max_{c_t, x_t} \mathbb{E} \sum_{t=0}^{+\infty} \beta^t (u(Rc_t + x_t) - px_t)
\]
subject to

\[
\begin{align*}
z_{t+1} &= z_t - c_t + \tilde{y}_t, \\
c_t &\leq z_t, \\
c_t &\geq z_t - Z, \\
x_t &\geq 0.
\end{align*}
\]

Consider the optimal policy \( c^* (z; p_0) \) and \( x^* (z; p_0) \) at price \( p_0 \). This allocation is feasible at price \( p_1 \). Therefore,

\[
v(z, p_0) \leq \mathbb{E} \sum_{t=0}^{+\infty} \beta^t (u(Rc^* (z, p_0) + x^* (z, p_0)) - p_1 x^* (z, p_0)),
\]

whence

\[
v(z, p_1) = v(z, p_1).
\]

**Proof of Lemma 7:**

To prove this result I thus focus on two random variables \( \tilde{z}_1 \) and \( \tilde{z}_2 \) such that \( \tilde{z}_1 = \{-\varepsilon, \varepsilon, 0.5, 0.5\} \) and \( \tilde{z}_2 = \{-z, z, 0.5, 0.5\} \) with \( \varepsilon < z \). I define

\[
H(c_1, \sigma_1) = R\hat{u}(Rc_1) - \beta \mathbb{E}v'(z - c_1 + \bar{y} + \tilde{z}_1, \sigma_1).
\]

\( c_2 \) defined by \( R\hat{u}(Rc_2) - \beta \mathbb{E}v' (z - c_2 + \bar{y} + \tilde{z}_2, \sigma_2) = 0 \) is lower than \( c_1 \) (the social planner is prudent) if and only if \( H(c_2, \sigma_1) > 0 \).

\[
H(c_2, \sigma_1) = R\hat{u}(Rc_2) - \beta \mathbb{E}v'(z - c_2 + \bar{y} + \tilde{z}_1, \sigma_1)
= \beta \mathbb{E}v'(z - c_2 + \bar{y} + \tilde{z}_2, \sigma_2) - \beta \mathbb{E}v'(z - c_2 + \bar{y} + \tilde{z}_1, \sigma_1).
\]
This implies that
\[
\frac{2}{\beta} H (c_2, \sigma_1) = v' (z - c_2 + \gamma + \bar{\epsilon}, \sigma_2) - v' (z - c_2 + \gamma + \bar{\epsilon}, \sigma_2) + v' (z - c_2 + \gamma + \bar{\epsilon}, \sigma_2) - v' (z - c_2 + \gamma + \bar{\epsilon}, \sigma_1) + \\
+ v' (z - c_2 + \gamma - \bar{\epsilon}, \sigma_2) - v' (z - c_2 + \gamma - \bar{\epsilon}, \sigma_2) + v' (z - c_2 + \gamma - \bar{\epsilon}, \sigma_2) - v' (z - c_2 + \gamma - \bar{\epsilon}, \sigma_1).
\]
I assume that \( \Delta \bar{\epsilon} = \bar{\epsilon} - \bar{\epsilon} \) is small so that I approximate \( v' (z - c_2 + \gamma + \bar{\epsilon}, \sigma_2) - v' (z - c_2 + \gamma + \bar{\epsilon}, \sigma_2) \) with \( \Delta \bar{\epsilon} \frac{\partial^2 v (z - c_2 + \gamma + \bar{\epsilon}, \sigma_2)}{\partial \sigma \partial z} \) and \( v' (z - c_2 + \gamma + \bar{\epsilon}, \sigma_2) - v' (z - c_2 + \gamma + \bar{\epsilon}, \sigma_1) \) by \( \Delta \sigma \frac{\partial^2 v (z - c_2 + \gamma + \bar{\epsilon}, \sigma_2)}{\partial \sigma \partial z} \). This implies
\[
\frac{2}{\beta} H (c_2, \sigma_1) = \Delta \bar{\epsilon} \frac{\partial^2 v (z - c_2 + \gamma + \bar{\epsilon}, \sigma_2)}{\partial z^2} + \Delta \sigma \frac{\partial^2 v (z - c_2 + \gamma + \bar{\epsilon}, \sigma_2)}{\partial z^2} - \\
\Delta \bar{\epsilon} \frac{\partial^2 v (z - c_2 + \gamma - \bar{\epsilon}, \sigma_2)}{\partial z^2} + \Delta \sigma \frac{\partial^2 v (z - c_2 + \gamma - \bar{\epsilon}, \sigma_2)}{\partial z^2}.
\]
The social planner is prudent if and only if
\[
\Delta \bar{\epsilon} \left( \frac{\partial^2 v (z - c_2 + \gamma + \bar{\epsilon}, \sigma_2)}{\partial z^2} - \frac{\partial^2 v (z - c_2 + \gamma - \bar{\epsilon}, \sigma_2)}{\partial z^2} \right) \geq - \Delta \sigma \left( \frac{\partial^2 v (z - c_2 + \gamma + \bar{\epsilon}, \sigma_2)}{\partial \sigma \partial z} + \frac{\partial^2 v (z - c_2 + \gamma - \bar{\epsilon}, \sigma_2)}{\partial \sigma \partial z} \right).
\]

**Proof of Lemma 8:**
I successively successively prove the four assertions.

1. Suppose there exists \( z_0 > 0 \) such that \( c_0 (z_0) = 0 \). This implies that \( y_0 \geq 0 \). The FOC of the maximization program leads to
   \[
   Ru' (Rc_0) = \beta [q \mathbb{E} \nu_0' (z_0 - c_0 + \tilde{y}) + (1 - q) \mathbb{E} \nu_1' (z_0 - c_0 + \tilde{y})] - y_0, \\
   \leq \beta [q \mathbb{E} \nu_0' (z_0 - c_0 + \tilde{y}) + (1 - q) \mathbb{E} \nu_1' (z_0 - c_0 + \tilde{y})], \\
   < q \mathbb{E} \nu_0' (z_0) + (1 - q) \mathbb{E} \nu_1' (z_0) \text{ because } v \text{ is concave}, \\
   = qRu' (c_0) + (1 - q) Ru' (Rc_1 + \bar{W}) \text{ because of the envelope theorem}.
   \]
   This leads to \( Rc_0 = 0 > Rc_1 + \bar{W} \) what is not possible since \( c_1 > 0 \). Therefore, the constraint \( c_0 (z) \geq 0 \) never binds for strictly positive \( z \) and \( y_0 = 0 \).

2. Suppose \( z_0 \) is strictly positive and \( c_1 (z_0) = 0 \). The FOC of the maximization program gives
   \[
   Ru' (Rc_1 + \bar{W}) < q \mathbb{E} \nu_0' (z_0 + \tilde{y}) + (1 - q) \mathbb{E} \nu_1' (z_0 + \tilde{y}), \\
   \leq qRu' (Rc_0) + (1 - q) Ru' (Rc_1 + \bar{W}).
   \]
   This leads to \( Rc_0 (z_0) < \bar{W} \).

3. I suppose \( c_1 (z) = z \) and \( c_0 (z) < z \).
   \( c_1 (z) = z \) implies that \( Ru' (Rz + \bar{W}) = \beta [q \mathbb{E} \nu_0' (\tilde{y}) + (1 - q) \mathbb{E} \nu_1' (\tilde{y})] + \lambda_1 \). \( c_0 (z) < z \) implies that
   \[
   Ru' (Rc_0) = \beta [q \mathbb{E} \nu_0' (z - c_0 + \tilde{y}) + (1 - q) \mathbb{E} \nu_1' (z - c_0 + \tilde{y})], \\
   \leq \beta [q \mathbb{E} \nu_0' (\tilde{y}) + (1 - q) \mathbb{E} \nu_1' (\tilde{y})], \\
   = Ru' (Rz + \bar{W}) - \lambda_1, \\
   \leq Ru' (Rz + \bar{W}).
   \]
The concavity of function \( u \) implies that \( Rc_0 \geq Rz + \bar{W} \), what cannot happen. Therefore, there is a contradiction and \( c_0 (z) = z \).

4. As for the previous result, suppose that \( c_0 (z) = z - \bar{Z} \) and \( c_1 (z) > z - \bar{Z} \). The equality gives
   \[
   Ru' (Rz - R\bar{Z}) = \beta [q \mathbb{E} \nu_0' (\bar{Z} + \tilde{y}) + (1 - q) \mathbb{E} \nu_1' (\bar{Z} + \tilde{y})] - \nu_0. \]
The inequality implies that
   \[
   Ru' (Rc_1 + \bar{W}) = \beta [q \mathbb{E} \nu_0' (z - c_0 + \tilde{y}) + (1 - q) \mathbb{E} \nu_1' (z - c_0 + \tilde{y})], \\
   > \beta [q \mathbb{E} \nu_0' (\bar{Z} + \tilde{y}) + (1 - q) \mathbb{E} \nu_1' (\bar{Z} + \tilde{y})], \\
   = Ru' (Rz - R\bar{Z}) + \nu_0, \\
   > Ru' (Rz - R\bar{Z}).
   \]
This leads to \( \bar{W} < 0 \), a contradiction. Therefore, \( c_1 (z) = z - \bar{Z} \). \( \square \)
Proof of Proposition 3:
Concerning the first result, suppose by contradiction there exists \( z_0 \) such that \( c_0(z_0) < c_1(z_0) \).
When neither constraint is binding and by the concavity of \( u \), this implies that: \( Ru' (Rc_0(z_0)) > Ru' (Rc_1(z_0) + W) \).
Depending on the first order conditions:
\[
q [E v'_0 (z_0 - c_0(z_0) + \tilde{y}) - E v'_0 (z_0 - c_1(z_0) + \tilde{y})] > (1 - q) [E v'_1 (z_0 - c_1(z_0) + \tilde{y}) - E v'_1 (z_0 - c_0(z_0) + \tilde{y})].
\]
As \( c_0(z_0) < c_1(z_0) \) and as \( z \mapsto v'(z;0) \) is a decreasing function, the left hand side is strictly negative.
Similarly, the right hand side is strictly positive and this leads to a contradiction.

When one of the constraint is binding, I know according to the results of Lemma 8 that \( c_1(z) \leq c_0(z) \).
Concerning the second point, suppose there exists \( z_1 \) such that \( Rc_0(z_1) > Rc_1(z_1) + W \).
When neither constraint is binding, this implies that: \( Ru' (Rc_1(z_1) + W) > Ru' (Rc_0(z_1)) \), and therefore:
\[
(1 - q) [E v'_0 (z_1 - c_1(z_1) + \tilde{y}) - E v'_0 (z_1 - c_0(z_1) + \tilde{y})] > q [E v'_1 (z_1 - c_0(z_1) + \tilde{y}) - E v'_1 (z_1 - c_1(z_1) + \tilde{y})].
\]
Once more, the assumptions imply that the left hand side is strictly negative and the right hand side strictly positive, there is a contradiction.

I consider rapidly the cases where one of the constraints is binding:
- If \( c_1(z) = 0 \), then according to the previous lemma, \( Rc_0(z) < W \).
- If \( c_0(z) = z \) and \( c_1(z) < z \), then the FOC lead to
\[
Ru' (Rc_1 + W) < \beta [q E v'_0 (\tilde{y}) + (1 - q) E v'_1 (\tilde{y})] = Ru' (Rc_0) - \lambda_0 < Ru' (Rc_0)
\]
Therefore, \( Rc_1 + W > Rc_0 \).
- If \( c_1(z) = z - Z \) and \( c_0(z) > z - Z \), a similar reasoning leads to the result. \( \square \)